Proceedings of the American Control Conference
Philadelphia, Pennsylvania • June 1998

Design of Optimal Mixed H_2/H_∞ Static State Feedback Controllers*

B. Halder, B. Hassibi and T. Kailath †
Stanford University, CA 94305

Abstract

Despite the recent advances in robust control theory, the robust performance problem formulated in the mixed H_2/H_∞ framework largely remains an open problem. In this approach, one seeks a controller that minimizes the H_2 norm of a closed-loop map over all admissible controllers while satisfying an H_∞ constraint on another closed-loop map. Unlike the optimal H_2 problem or the γ-level sub-optimal H_∞ problem, the mixed H_2/H_∞ problem does not have a readily computable solution. In this paper we restrict consideration to *static* state feedback controllers and propose an efficient iterative algorithm for computing the optimal H_2/H_∞ solution.

1. Introduction

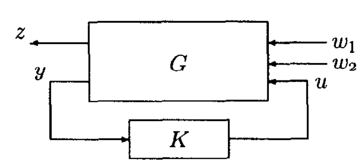
Optimal performance and robustness are arguably the two most desirable properties of any controller. The mixed H_2/H_∞ approach to controller design provides a framework to integrate these two features into a single controller. The mixed H_2/H_∞ problem can be formulated as a constrained optimal control problem (COCP) that yields a mixed controller with robust H_2 performance. However, the mixed problem can also be thought of as a way to improve the H_2 performance by exploiting the non-uniqueness of the suboptimal H_∞ solutions.

As it turns out, the mixed H_2/H_∞ problem, though easy to state and motivate, is surprisingly hard to solve analytically. However, various related problems have been suggested and algorithms have been proposed to obtain suboptimal solutions. One approach is to replace the H_2 cost by a suitable upper bound. The maximum entropy method of Glover and Mustafa [4], the "auxiliary cost" cost method of Bernstein and Haddad [2] and the method proposed by Doyle et al. [9] fall into this category. In [5] Khargonekar and Rotea developed a computationally efficient convex formulation for the state feedback case using the same auxiliary cost function as in [2]. However, numerical results reveal two significant drawbacks: i) the true H_2 norm of the optimal modified mixed H_2/H_∞ solution can even be worse than that of the central solution and ii) the modified mixed H_2/H_∞ solution fails to achieve the optimal H_2 performance even when the specified H_∞ norm bound is larger than the H_∞ norm of the H_2 optimal solution [1]. Therefore, minimization of the upper bound may not be an effective way to reduce the true H_2 norm of the closed-loop system. This motivated us to reconsider the original mixed problem with the true H_2 norm.

Unlike the optimal H_2 problem or suboptimal H_∞ problem, the mixed problem is not guaranteed to have a static feedback solution [6]. On the other hand, the key difficulty in considering dynamic feedback is that the pure mixed problem may not have a bounded order solution even for finite order plants [3]. It seems unlikely that such infinite-order solutions can be obtained by using some finite dimensional optimization technique. Hence, to obtain a computable solution, it is reasonable to restrict the search to static state feedback controllers at the cost of some performance loss. This leads to a meaningful finite dimensional (non-convex) optimization problem and we develop an efficient iterative algorithm to solve the optimization problem. Each iteration of the proposed algorithm consists of three easily solvable subproblems: i) an analytic centering problem for a linear matrix inequality (LMI), ii) a semi-definite programming (SDP) problem and iii) an one-dimensional line search. The main features of the algorithm is a guaranteed "descent" at each step. This ensures that unlike the method proposed in [5] the proposed algorithm always yields a state feedback controllers that out performs the central controller.

In Sec. 2., we present the system description and the problem definition along with some relevant results from the H_2 and H_∞ theories. The BMI representation and the decomposition of the set of all controllers satisfying the H_∞ constraint is studied in Sec. 3. After discussing some relevant properties of the cost function in Sec. 4., we outline the proposed algorithm in Sec. 5. After presenting an example in Sec. 6., we conclude the paper in Sec. 7.

2. The system model



*This work was supported in part by DARPA through the Department of Air Force under contract F49620-95-1-0525-P00001 and by the Joint Service Electronics Program at Stanford under contract DAAH04-94-G-0058-P00003.

†Email addresses (in order): bjt@isl.stanford.edu, hassibi@isl.stanford.edu, and tk@isl.stanford.edu.

Consider a linear time invariant plant (G):

$$G := \begin{cases} \dot{x}(t) &= Ax(t) + B_1 w_1 + B_2 w_2(t) + Bu(t) \\ z(t) &= Cx(t) + Du(t) \\ y(t) &= x(t) \end{cases} \tag{1}$$

We make the following assumptions:
A1) (A, B) is stabilizable.
A2) D has full column rank.
A3) $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full column rank for all $\omega \in \mathcal{R}$.
The assumption A1 guarantees the existence of a controller that stabilizes the plant G. The last two assumptions A2-3 are standard and are made to ensure the regularity of the H_2 problem. For a uncluttered presentation we also assume that
A4) $D^T[C\ D] = [0\ I]$
which can be achieved by a feedback transformation without loss of generality.

In this paper, we restrict ourselves to static state feedback laws of the form $u(t) = Ky(t)$, yielding the closed-loop systems of the form

$$G_{cl} := \begin{cases} \dot{x}(t) &= (A + BK)x(t) + B_1 w_1(t) + Bw_2(t) \\ z(t) &= (C + DK)x(t). \end{cases}$$

A controller K is *addmisible* if $A_k \triangleq A + BK$ is stable.

Problem 1 *Given an achievable H_∞ bound γ, find an internally stabilizing static state feedback law, $u(t) = Kx(t)$, that satisfies*

$$\min_K \|T_{zw_2}\|_2, \quad \textit{subject to} \quad \|T_{zw_1}\|_\infty \le \gamma, \tag{2}$$

where and $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the H_2 and H_∞ norms, respectively and T_{zw_i}, $i = 1, 2$, denote the closed-loop maps from w_i to z corresponding to a state feedback matrix K, i.e., $G_{cl} = [T_{zw_1}\ T_{zw_2}]$.

2.1. Results from H_2 and H_∞ theory

The H_2 norm of the closed-loop map T_{zw_2} is given by

$$\|T_{zw_2}\|_2 = \{tr(B_2^T Y_k B_2)\}^{1/2}$$

where Y_k is the observability Gramian of the pair $(A_k, C + DK)$. For the system G, the optimal H_2 state feedback is given by $K_2 \triangleq -B^T X_2$, where X_2 is the stabilizing solution of the following algebraic Riccati equation:

$$X_2 A + A^T X_2 - X_2 BB^T X_2 + C^T C = 0. \tag{3}$$

Now by the bounded real lemma, we know that for an admissible K, the closed-loop map T_{zw_1} has H_∞ norm $\le \gamma$, if and only if, there exists an $X \ge 0$ such that

$$XA_k + A_k^T X + \gamma^{-2} X B_1 B_1^T X + K^T K + C^T C \le 0. \tag{4}$$

The existence of such a state feedback matrix K is equivalent to the existence of a positive semidefinite matrix X that satisfies the Riccati inequality,

$$R(X) \triangleq -XA - A^T X + X(BB^T - \gamma^{-2} B_1 B_1^T)X - C^T C \ge 0. \tag{5}$$

Given an X satisfying (5), $K = -B^T X$ is one such state feedback matrix satisfying (4). Moreover, the smallest such X is denoted by X_c and is the positive semidefinite solution to the Riccati equation

$$X_c A + A^T X_c - X_c(BB^T - \gamma^{-2} B_1 B_1^T)X_c + C^T C = 0. \tag{6}$$

The corresponding feedback matrix $K_c (= -B^T X_c)$ is known as the *central* solution.

Definition 1 *Let S_K be the set of static state feedback matrices that yields a internally stable closed-loop system with $\|T_{zw_1}\|_\infty \le \gamma$, i.e.,*

$$S_K \triangleq \{\textit{admissible } K \mid \|T_{zw_1}\|_\infty \le \gamma\}. \tag{7}$$

Problem 2 *The pure mixed problem described in Problem 1 is equivalent to the following constrained minimization problem:*

$$\min_{K \in S_K} J(K) \tag{8}$$

where $J(K) \triangleq tr(B_2^T Y_k B_2)$.

3. The Set S_K

Let S_X be the set of all $X \ge 0$ that satisfies (5), i.e.,

$$S_X \triangleq \{X \ge 0 \mid R(X) \ge 0\}. \tag{9}$$

Rewriting (4) as a BMI, we get

$$B(X,K) \triangleq \begin{bmatrix} -XA_k - A_k^T X - C^T C & (K^T\ \gamma^{-1} X B_1) & 0 \\ (K^T\ \gamma^{-1} X B_1)^T & I & 0 \\ 0 & 0 & X \end{bmatrix} \ge 0. \tag{10}$$

Hence, S_K is the set of admissible K that satisfies the BMI (10). For a fixed $X \ge 0$ the BMI reduces to a LMI that can be rewritten as

$$L_X(K) \triangleq \begin{bmatrix} -XA_k - A_k^T X - \gamma^{-2} X B_1 B_1^T X - C^T C & K^T \\ K & I \end{bmatrix} \ge 0. \tag{11}$$

For a given $X \ge 0$ the feasibility of the resulting LMI (11) is not guaranteed. However, we have the following result.

Lemma 1 *For a given $X \ge 0$ the LMI, $L_X(K) \ge 0$ as defined in (11) is feasible if and only if $X \in S_X$.*

For any $X \in S_X$, let $S_K(X)$ be the nonempty convex set of feedback matrices that satisfies $L_X(K) \ge 0$, then

$$S_K(X) = \{K = -B^T X + \Delta K \mid \Delta K^T \Delta K \le R(X)\}. \tag{12}$$

$S_K(X)$ is the largest set of feedback matrices for which the guarantee that $\|T_{zw_1}\|_\infty \le \gamma$ can be proven using the matrix X. The set S_K is essentially composed of all $S_K(X)$ as stated in the next lemma.

Lemma 2 *The set of feedback matrices that achieve* $\|T_{zw_1}\|_\infty \leq \gamma$ *can be expressed as a union of convex sets as follows:*

$$S_K = \bigcup_{X \in S_X} \{-B^T X + \Delta K \mid \Delta K^T \Delta K \leq R(X)\}.$$

4. Properties of $J(K)$

Lemma 3 *Over the set of admissible feedback matrices the objective function* $J(K)$ *is differentiable and the gradient* G_k *with respect to the matrix variable* K *is given by*

$$G_k \triangleq \frac{\partial J(K)}{\partial K} = 2(B^T Y_k + K)\Sigma_k, \quad (13)$$

where Σ_k *is the controllability Gramian of* (A_k, B_2). *Moreover, the second directional derivative along a direction* ΔK *is given by*

$$H(K, \Delta K) = 2tr(\Sigma_k(2B^T Y_k^{(1)} + \Delta K)^T \Delta K), \quad (14)$$

where $Y_k^{(1)}$ *satisfies a Lyapunov equation.*

Next, using the fact that all the stationary points satisfy the equation $G_k = 0$, we derive the following characterization of all the stationary points of $J(K)$.

Theorem 1 *K is a stationary point of $J(K)$, if and only if, $K = -B^T Y + L$ for some L in the uncontrollable subspace of the pair $(A - BB^T Y, B_2)$, and Y is the stabilizing solution of the following Riccati equation:*

$$YA + A^T Y - YBB^T Y + L^T L + C^T C = 0.$$

Recall that the H_2 solution is unique if in addition to the stabilizability of the pair (A, B), the matrix B_2 has full row rank [7]. However, if we assume the controllability of the pair (A, B), then the assumption on the row rank of B_2 can be relaxed as follows.
A5) Let (A, B) be a controllable pair and $\mathcal{N}(B_2) \subset \mathcal{N}(B)$, where $\mathcal{N}(\cdot)$ denotes the null space of a matrix.

Lemma 4 *Assume A5 holds, then $J(K)$ has a unique stationary point at K_2 over all admissible controllers. Moreover, the stationary point is also the global minimum.*

The assumption on the null space prohibits the use of any noise free actuators.

Lemma 5 *Assume A1-5 hold. Let K be a minima of $J(K)$ over any closed set of admissible controllers and let K be an interior point. Then, $K = K_2$.*

5. Development of the Algorithm

The cost function $J(K)$, though nonlinear, is a well-behaved smooth function of K with easily computable gradient and second order directional derivative. However, due to the non-convexity of the constraint set S_K, the problem of enforcing the H_∞ constraint at each step is a non-trivial one. Moreover, even to verify that a given controller K satisfies the H_∞ constraint we need to solve a quadratic matrix Riccati equation. To tackle this difficulty, given a $K \in S_K$, we first replace the non-convex constraint set S_K by a convex set Φ_K that satisfies three properties: P1) Φ_K is guaranteed to contain a controller that yields a lower value of the cost function, P2) Φ_K is a subset of the constraint set S_K, P3) the set is easy to compute and has a simple parameterization. Such replacement yields a sub-optimal problem that is substantially easy to solve. Second, we minimize the cost function $J(K)$ over Φ_K to obtain a controller that yields a lower $\|T_{zw_2}\|_2$.

This process is repeated until no such Φ_K can be constructed around the current point. This can happen for two reasons: i) the current point is on the boundary of the constraint set S_K or, ii) we have reached a stationary point. Recall that, under assumption A5 convergence to the stationary point implies convergence to the global minimum.

5.1. Construction of the set $\Phi(K)$

Given a controller $K \in S_K$ we identify the set of positive semi-definite matrices that prove that $K \in S_K$, and denote it by $S_X(K)$. Hence,

$$S_X(K) = \{X \mid L_K(X) \geq 0\}, \quad (15)$$

where

$$L_K(X) \triangleq \begin{bmatrix} -XA_k - A_k^T X - K^T K & \gamma^{-1} X B_1 & 0 \\ \gamma^{-1} B_1^T X & I & 0 \\ 0 & 0 & X \end{bmatrix}.$$

Clearly, $S_X(K)$ is a subset of S_X. Therefore, associated with each $X \in S_X(K)$ there is a non-empty convex set of controllers $S_K(X)$ (see (12)) that are guaranteed to satisfy the H_∞ constraint. From $S_X(K)$, we select the X that yields the "largest" set of controllers $S_K(X)$. This "largest" set of controllers $S_K(X)$ will be the Φ_K of our choice.

Lemma 6 *For a given controller K that is not a saddle point of the function $J(K)$ and lies in the interior of S_K, there always exists a Φ_K satisfying the first two properties P1 and P2.*

For a given X, the size of the set $S_K(X)$ is solely determined by the matrix $R(X)$. Now, determinant of a matrix is a reasonable indicator of its size and we select the $X \in S_X(K)$ that yields the $R(X)$ with maximum determinant. Therefore, given a controller K, the $X \in$

$S_X(K)$ that yields the largest set of controllers can be found by solving the following problem:

$$\max_{X \in S_X(K)} \log\det R(X).$$

Unfortunately, $\log\det R(X)$ is not a concave function of X over the set $S_X(K)$. This motivated us to formulate a closely related convex problem by replacing $R(X)$ by a lower bound as follows:

$$\max_{X\geq 0} \log\det W_k(X).$$

where $W_k(X) = R(X) - (B^TX+K)^T(B^TX+K)$.

Let,

$$X^* = \arg\max_X \log\det W_k(X), \quad (16)$$

then we define the set Φ_K as

$$\Phi_K \triangleq \{K = -B^TX^* + \Delta K \mid \Delta K^T\Delta K \leq R(X^*)\}. \quad (17)$$

Note that the maximization problem (16) is well-defined for all K in the interior of S_K. We would like to point out that for the purpose of numerical implementation of the algorithm finding the exact maximum is not required and problem (16) can be replaced by a much simpler problem of the form

$$\max\ \log\det W_k(X) + \alpha \log\det X, \quad (18)$$

with some positive scalar $\alpha \ll 1$.

5.2. Minimization of $J(K)$ over Φ_K

Lemma (6) guarantees the existence of a controller with improved H_2 performance unless the given controller K is a stationary point. We obtain such a controller by minimizing the cost function $J(K)$ over Φ_K, i.e.,

$$K = \arg\min_{K\in\Phi_K} J(K). \quad (19)$$

Note that this optimization problem differs from the original one (8) only in the constraint set. Since, we have replaced the non-convex constraint set S_K with a convex one Φ_K, the resulting minimization problem is much easier to solve. Now, we present a feasible direction based method to solve the constraint minimization problem (19).

The set of all feasible directions of Φ_K at K, $\mathcal{F} = \{\Delta K | \Delta K \neq 0, \text{ and } K + \lambda\Delta K \in \Phi_K \text{ for all } \lambda \in (0,\delta) \text{ for some } \delta > 0\}$. The set of all descent directions of $J(K)$ at K, $\mathcal{D} = \{\Delta K | tr(G(K)\Delta K^T) < 0\}$. Given a feasible point K, a direction ΔK is a *descent feasible direction* if $\Delta K \in \mathcal{F}\cap\mathcal{D}$. Once such a direction is obtained a line search is performed to find a minima within the feasible set. This leads to a new improved feasible point. This process is continued until we reach a point where there is no descent feasible direction.

We now describe a method to construct a descent feasible direction for $J(K)$ over the set Φ_K. Given a feasible point K, we approximate $J(K+\Delta K)$ around K by a quadratic function of ΔK as $J(K+\Delta K) \approx J(K) + Q_k(\Delta K)$, where

$$Q_k(\Delta K) = tr\left[\Sigma_k(\Delta K^T\Delta K + (K+B^TY_k)^T\Delta K + \Delta K^T(K+B^TY_k))\right]. \quad (20)$$

Now, let

$$\Delta K^* = \arg\min_{\Delta K} Q_k(\Delta K), \text{ such that } K+\Delta K \in \Phi_K. \quad (21)$$

The above convex problem can be converted to a semidefinite programming (SDP) problem, and hence, can be easily solved (see, [8]).

Lemma 7 *If $\Delta K = 0$ is not a minimizer of $Q_k(\Delta K)$ then any nonzero direction ΔK^* as defined in (21) is a decent feasible direction for the function $J(K)$ at a feasible point K over the set Φ_K. Moreover, $\Delta K = 0$ is a minimizer, if and only if, there is no descent feasible direction at K, i.e., $\mathcal{F}\cap\mathcal{D} = \phi$.*

Once a decent feasible direction ΔK^* is available, the next update is obtained by the following line search:

$$\min_{\lambda\in(0\ 1]} J(K+\lambda\Delta K^*). \quad (22)$$

Algorithm A: Feasible direction method to minimize $J(K)$ over Φ_K

1 *Feasible direction*: Given a feasible K, obtain a descent feasible direction ΔK^* by solving (21). If $Q_k(\Delta K^*) = 0$ then stop; otherwise, go to Step 2.

2 *Line search*: Perform a line search along the direction ΔK^* as described in (22). Let $\lambda^* \in (01]$ is a solution of (22), then set $K := K + \lambda^*\Delta K^*$.

3 *Stopping Criterion*: If the improvement in the cost function is lower than some pre-specified tolerance, i.e. $(J(K) - J(K+\lambda^*\Delta K^*))/J(K) < tol$ then stop. Otherwise repeat Step 1.

Lemma 8 *The feasible direction algorithm (**A**) described above to minimize $J(K)$ over Φ_K is a descent algorithm and converges to a point with no descent feasible direction.*

5.3. The Algorithm

1 *The Starting Point*: For a given γ that is achievable find the central controller K_c.

2 *Construction of the set Φ_K*: For a given K in the strict interior of the set S_K, i.e., $\|T_{zw_1}\|_\infty < \gamma$ find an $X \geq 0$ that solves the maxdet problem as described in (16) and calculate $R(X)$ using (5).

3 *Reduction of H_2 norm*: Obtain a new controller by solving the minimization problem (19).

4 *The Stopping Criteria*: Repeat Step 2 and 3 until: i) the boundary of S_K is reached, i.e., $\|T_{zw_1(K)}\|_\infty = \gamma$ or ii) $G_k = 0$.

6. Numerical Example

For the purpose of illustration we present a numerical example. We consider a 4-th order LTI system. For this example the matrices B_2 and B do not satisfy the assumption (A5). This choice was made to stress that though the assumption (A5) is helpful to prove certain existence results, it is not necessary for the convergence of the algorithm. For this system, the optimal H_∞ norm of T_{zw_1} is approximately $4.7525(\triangleq \gamma_o)$ and the H_2 norm of T_{zw_2} for the central controller is 40.1891. On the other hand, the optimal H_2 controller yields $\|T_{zw_1}\|_\infty = 9.2453(\triangleq \gamma_2)$ and $\|T_{zw_2}\|_2 = 14.3979$. In Figure (1) we plot the H_2 norm of the closed-loop map T_{zw_2} achieved by the central controller and the optimal mixed controller as a function of the parameter γ. The figure shows that the mixed controller always yields a closed-loop map with smaller H_2 norm. In Figures (2) we plot the value of the objective function for $\gamma = 6.2$ as a function of iteration number. Initially the improvement in the cost is large and the decrease in the value of the cost function slows down as we approach the boundary of the set.

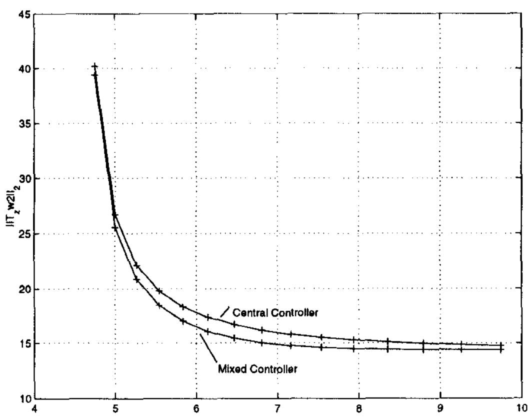


Figure 1: $\|T_{zw_2}\|_2$ achieved by the central and the optimal mixed controllers as a function γ.

7. Conclusions

In this paper we propose an iterative algorithm for efficient computation of optimal mixed H_2/H_∞ static state feedback controllers. By restricting the controller to the class of static feedbacks, we are able to formulate the mixed H_2/H_∞ problem as a finite dimensional nonlinear minimization problem. The proposed algorithm converts this difficult nonlinear programming problem into a series of convex subproblems, each of which can be readily solved. As a result we obtain a descent algorithm that is guaranteed to yield a controller with an improved H_2 performance compared to the H_2 performance of the central solution of the suboptimal H_∞ problem. We also provide

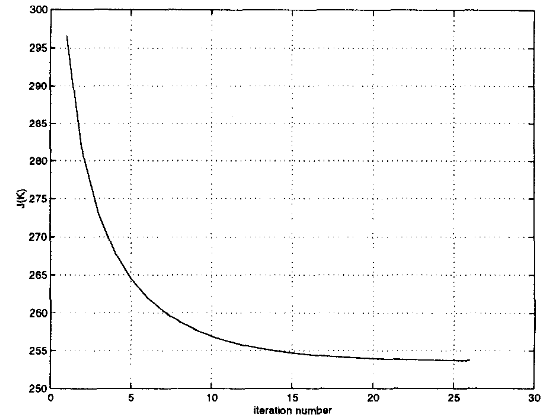


Figure 2: The value of cost function $J(K)$ after each iteration (γ=6.2).

a numerical example to illustrate the improvement in the H_2 performance, over the H_2 performance obtained by the central controller.

References

[1] Baeyens, E., and P. P. Khargonekar (1994), Some examples in mixed H_2/H_∞ control, In *Proc. ACC*, June, pp. 1608-1612.

[2] Bernstein, D. S. and W. M. Haddad (1989), LQG control with an H_∞ performance bound: A Riccati equation approach, *IEEE Trans. on AC*, vol.34, no.3, pp. 293-305.

[3] Foias, C., et al. (1995), On H^2 minimization for the Caratheodory-Schur interpolation problem, *Integral Equations and Operator Theory*, vol.21, no.1, pp. 24-32.

[4] Glover, K. and D. Mustafa (1989), Derivation of the maximum entropy H_∞-controller and a state-space formula for its entropy, *Int. J. Cont.*, vol. 50, pp. 899-916.

[5] Khargonekar, P. P.,and M. A. Rotea (1992), Mixed H_2/H_∞ control: A convex optimization approach, In *IEEE Trans. on AC*, vol. 36, no.7, pp. 824-937.

[6] Rotea, M. A., and P. P. Khargonekar (1992), H_2-optimal control with an H_∞-constraint: The state feedback case, *Automatica*, vol.27, no.2, pp. 307-316.

[7] Saberi, A., P. Sannuti and B. M. Chen (1995) *H_2 Optimal Control*, Prentice-Hall.

[8] Vandenberghe, L., and S. Boyd (1996), Semidefinite programming, *SIAM Review*, vol.38, no.1, pp. 49-95.

[9] Zhou, K., K. Glover, et al. (1994), Mixed H_2 and H_∞ performance objectives I and II, *IEEE Trans. on AC*, vol.39, no.8, pp. 1564-1587.