Equalization with an H^∞ Criterion[1]

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Abstract — **We study the problem of linear equalization from an H^∞ point of view and parametrize, in closed form, all possible H^∞-optimal equalizers. The results indicate an interesting dichotomy between minimum phase and non-minimum phase channels: for minimum phase channels the best causal equalizer performs as well as the best noncausal equalizer, whereas for non-minimum phase channels, causal equalizers cannot reduce the estimation error bounds from their a priori values.**

I. The Linear Equalization Problem

The equalization problem studied in this paper is depicted in Fig. I. Here, $\{u_i\}$ is an unknown sequence (the *transmitted sequence*), $\{v_i\}$ is an unknown additive disturbance sequence, $\{y_i\}$ is a known *observations* sequence, and $H(z)$ is a known causal linear time-invariant (LTI) communications channel. The goal is to design the linear time-invariant system $K(z)$ (the so-called *equalizer*) so as to estimate $\{u_i\}$ from $\{y_i\}$, where the estimated sequence is denoted by $\{\hat{u}_i\}$.

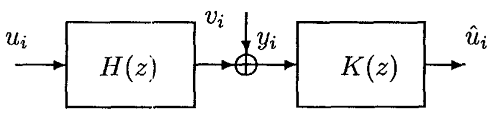

Figure 1: The linear equalization problem.

In the H^∞ framework, no specific statistical structure is assumed for the unknown signals $\{u_i\}$ and $\{v_i\}$, and the goal is to optimize the worst-case performance of the equalizer, which is taken to be the maximum energy gain from the disturbances, $\{u_i, v_i\}$, to the estimation errors $\{u_i - \hat{u}_i\}$, *i.e.*,

$$\sup_{u,v\in l^2-\{0\}} \frac{\|u-\hat{u}\|^2}{\|u\|^2+\|v\|^2}, \tag{1}$$

where $\|a\|^2 \triangleq \sum_j |a_j|^2$ and l^2 denotes the space of square-summable sequences. Since the above criterion deals with the worst-case disturbances, $\{u_i, v_i\}$, it guarantees a certain amount of robustness with respect to uncertainty on the statistics of $\{u_i, v_i\}$, and on the model of the channel $H(z)$, itself.

II. The H^∞ Solution

In minimizing the criterion of (1), over choice of the equalizer $K(\cdot)$, two cases can be envisioned: the noncausal case, where the equalizer has access to future values of the observations sequence $\{y_i\}$, and the causal case, where the equalizer has access only to current and past values of $\{y_i\}$. We now consider each separately.

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Theorem 1 (Noncausal H^∞-Optimal Equalizer) *The min-max energy gain for the problem*

$$\inf_{K(\cdot)} \sup_{u,v\in l^2-\{0\}} \frac{\|u-\hat{u}\|^2}{\|u\|^2+\|v\|^2} \triangleq \gamma_s^2,$$

is given by $\gamma_s = \sup_{\omega\in[0,2\pi]} \frac{1}{1+|H(e^{j\omega})|^2}$. *Moreover, one noncausal H^∞-optimal solution is given by the noncausal MMSE equalizer,* $K_s(z) = \frac{H^*(z^{-*})}{1+H(z)H^*(z^{-*})}$.

Theorem 2 (Causal H^∞-Optimal Equalizer) *Consider the problem*

$$\inf_{\text{causal } K(\cdot)} \sup_{u,v\in l^2-\{0\}} \frac{\|u-\hat{u}\|^2}{\|u\|^2+\|v\|^2} \triangleq \gamma_c^2.$$

(i) If the channel $H(z)$ is minimum phase, i.e., if $H^{-1}(z)$ is analytic in $|z| \geq 1$, then $\gamma_c = \gamma_s$, with γ_s as in Thm. 1.

(ii) If the channel $H(z)$ is non-minimum phase, i.e., if $H^{-1}(z)$ is not analytic in $|z| \geq 1$, then $\gamma_c = 1$.

Further, let $h_0 = H(\infty)$, and R_Δ and the monic ($\Delta(\infty) = 1$) and minimum phase $\Delta(z)$ be given by the spectral factorization $\Delta(z)R_\Delta\Delta^*(z^{-*}) = \gamma_c^2 H(z)H^*(z^{-*})/(1-\gamma_c^2) - 1$. Then, in the minimum phase case, all possible H^∞-optimal equalizers are given by

$$K(z) = \frac{(1-\gamma_c^2)\left(1+\sqrt{\frac{1-\gamma_c^2}{R_\Delta}}\cdot\frac{S(z)}{h_0}\right)}{H(z) - \frac{1-\gamma_c^2}{h_0}R_\Delta\Delta(z) + (H(z)-\Delta(z)h_0)\sqrt{\frac{1-\gamma_c^2}{R_\Delta}}\cdot\frac{S(z)}{h_0}},$$

where $S(\cdot)$ is any causal contraction, *i.e.*, $S(z)$ is analytic in $|z| \geq 1$, and $|S(e^{j\omega})| \leq 1$, for all $\omega \in [0, 2\pi]$.

Note that $\gamma_c = 1$ is the maximum energy gain obtained from choosing $K(\cdot) = 0$ (simply replace $\hat{u}_i = 0$ in (1)). Thus, causal equalization of non-minimum phase channels is not possible. Therefore of related interest is the question of the minimum amount of delay (as opposed to the infinite delay required in the noncausal case) necessary to equalize a non-minimum phase channel.

Theorem 3 (H^∞ Equalization with Delay) *Define $K^d(z) = z^{-d}K(z)$, for some $d > 0$, and consider the problem*

$$\inf_{\text{causal } K^d(\cdot)} \sup_{u,v\in l^2-\{0\}} \frac{\|u-\hat{u}\|^2}{\|u\|^2+\|v\|^2} \triangleq \gamma_d^2.$$

Then $\gamma_d < 1$ if, and only if, d is greater than or equal to the number (counting multiplicities) of non-minimum phase (outside the unit circle) zeros of $H(z)$.

Proofs of these results, in the more general multiple-input multiple-output setting, can be found [1].

References

[1] B. Hassibi, A.T. Erdogan and T. Kailath, "Equalization with an H^∞ criterion", submitted to the *IEEE Trans. on Inf. Thy.*, 1997.