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On Robust Two-Block Problems[1]

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Abstract

In this paper we consider the following robust two-block problem that arises in estimation and in full-information control: minimize the worst-case H^∞ norm of a two-block transfer matrix whose elements contain H^∞-norm-bounded modeling errors. We show that, when the underlying systems are single-input/single-output, and if the modeling errors are "small enough", then the robust two-block problem can be solved by solving a one-dimensional family of appropriately-weighted "modeling-error-free" two-block problems. We also study the consequences of this result to a robust tracking problem, where the optimal solution can be explicitly found.

1 Introduction

In this paper we study the robust two-block problem

$$\inf_{Q(\cdot)\in H^\infty} \sup_{(\Delta P_1(\cdot),\Delta P_2(\cdot))\in B_\delta^\infty} \|T_Q(P_1+\delta P_1, P_2+\delta P_1)\|_\infty , \tag{1}$$

where

$$T_Q(P_1+\delta P_1, P_2+\delta P_1) =$$

$$\begin{bmatrix} P_1(z)+\Delta P_1(z)+(P_2(z)+\Delta P_2(z))\,Q(z) \\ Q(z) \end{bmatrix},$$

and

$$B_\delta^\infty = \left\{(\Delta P_1(\cdot),\Delta P_2(\cdot)), \left\|\begin{bmatrix} \Delta P_1(\cdot) & \Delta P_2(\cdot) \end{bmatrix}\right\|_\infty \le \delta\right\},$$

with $Q(\cdot)$, $P_1(\cdot)$, $P_2(\cdot)$, $\Delta P_1(\cdot)$ and $\Delta P_2(\cdot)$ all *scalar* functions in H^∞. This problem can be considered a robust version of the standard two-block problem

$$\inf_{Q(\cdot)\in H^\infty} \left\|\begin{bmatrix} P_1(z)+P_2(z)Q(z) \\ Q(z) \end{bmatrix}\right\|_\infty , \tag{2}$$

that arises in H^∞ full-information control, and in H^∞ estimation, which further allows one to consider possible modeling errors $\Delta P_1(\cdot)$ and $\Delta P_2(\cdot)$ for the nominal plants $P_1(\cdot)$ and $P_2(\cdot)$. In problem (1) both the objective and the modeling errors are measured in the H^∞ norm. Thus $\delta > 0$ is a measure of the modeling error allowed for in (1).

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2 Main Result

Problem (1) is, of course, a highly nonlinear problem and satisfactory solutions to date do not exist. In this paper, we show that for "small enough" modeling errors problem (1) can be solved by doing a one-dimensional search over the solution of a certain family of weighted standard two-block problems.

Thus, consider the solution to the following weighted two-block problem:

$$f(\epsilon) \triangleq \inf_{Q(\cdot)\in H^\infty} \left\|\begin{bmatrix} \sqrt{1+\epsilon}\,(P_1(z)+P_2(z)Q(z)) \\ \sqrt{1+(1+\frac{1}{\epsilon})\delta^2 Q(z)} \end{bmatrix}\right\|_\infty^2 + (1+\frac{1}{\epsilon})\delta^2, \quad \epsilon > 0. \tag{3}$$

The above problem can be readily solved for any value of ϵ (say, by using Riccati-based techniques when $P_1(\cdot)$ and $P_2(\cdot)$ are rational), and so $f(\epsilon)$ is easy to compute. Moreover, it can be shown that $f(\cdot)$ is, in general, a nonconvex continuous function of ϵ. Suppose now that we perform a one-dimensional search over $\epsilon > 0$, and determine

$$\epsilon^* \triangleq \inf_{\epsilon>0} f(\epsilon). \tag{4}$$

Then we have the following result.

Theorem 1 (Robust Two-Block Problem) *There exists a $\bar\delta > 0$, such that for all $\delta < \bar\delta$, the solution to the robust two-block problem (1) can be found from the solution to the weighted two-block problem*

$$\inf_{Q(\cdot)\in H^\infty} \left\|\begin{bmatrix} \sqrt{1+\epsilon^*}\,(P_1(z)+P_2(z)Q(z)) \\ \sqrt{1+(1+\frac{1}{\epsilon^*})\delta^2 Q(z)} \end{bmatrix}\right\|_\infty^2 + (1+\frac{1}{\epsilon^*})\delta^2. \tag{5}$$

The above theorem states that if the modeling error is less than $\bar{\delta} > 0$, then the robust two-block problem (1) can be solved using a one-dimensional search over a family of weighted two-block problems. Moreover, the solution to (1) is the same as the solution to a certain weighted two-block problem, with optimal weighting determined by ϵ^*. This has rather interesting physical implications since it states that the modeling errors $\Delta P_1(\cdot)$ and $\Delta P_2(\cdot)$ can be dealt with by appropriately weighting the modeling-error-free two-block problem (2).

Of course, this result raises several issues:

- How does Theorem 1 generalize to matrix plants?
- How does Theorem 1 generalize to four-block problems?
- How large is the value of $\bar{\delta}$ in Theorem 1?

Currently all three questions are open. To gain some insight into the third question, let us consider the robust tracking problem.

3 Robust Tracking

The H^∞ tracking problem corresponds to $P_1(z) = 1$ and $P_2(z) = -P(z)$, so that the robust tracking problem takes the form

$$\inf_{Q(\cdot)\in H^\infty} \sup_{\|\Delta P(\cdot)\|_\infty \le \delta} \left\| \begin{bmatrix} 1-(P(z)+\Delta P(z))\,Q(z) \\ Q(z) \end{bmatrix} \right\|_\infty . \tag{6}$$

The modeling-error-free tracking problem,

$$\inf_{Q(\cdot)\in H^\infty} \left\| \begin{bmatrix} 1-P(z)Q(z) \\ Q(z) \end{bmatrix} \right\|_\infty \triangleq \gamma_{opt}, \tag{7}$$

has been studied in [1], where it is shown:

- If $P(z)$ is minimum phase, then

$$\gamma_{opt} = \frac{1}{1+\min_{\omega\in[0,2\pi]} |P(e^{j\omega})|^2}. \tag{8}$$

- If $P(z)$ is minimum phase, then

$$\gamma_{opt} = 1. \tag{9}$$

Let us now return to problem (6). Clearly, if $P(z)$ is nonminimum phase, we can always obtain an H^∞ norm of unity in the objective cost by setting $Q(\cdot) = 0$. This is the same value obtained in the modeling-error-free case. Therefore, let us focus on the case where $P(z)$ is minimum phase.

Here we will have to distinguish between two cases:

(i) $\delta \ge \min_{\omega\in[0,2\pi]} |P(e^{j\omega})|^2 \triangleq p_{min}$. In this case, there exist modeling errors for which $P(\cdot)+\Delta P(\cdot)$ is non-minimum phase. Thus here the best choice is $Q(\cdot) = 0$, which results in an H^∞ norm of unity.

(ii) $\delta < \min_{\omega\in[0,2\pi]} |P(e^{j\omega})|^2 \triangleq p_{min}$. In this case, $P(\cdot) + \Delta P(\cdot)$ is always minimum phase.

Thus, clearly the case of interest is case (ii), above. The next result shows that for this case, the optimally-weighted two-block problem *always* solves (6).

Theorem 2 (Robust Tracking) *Consider problem (6) and suppose that*

$$\delta < \min_{\omega\in[0,2\pi]} |P(e^{j\omega})|^2 \triangleq p_{min}.$$

Then the solution to problem (6) is given by the solution to the problem,

$$\inf_{Q(\cdot)\in H^\infty} \left\| \begin{bmatrix} \sqrt{1+\epsilon^*}\,(1-P(z)Q(z)) \\ \sqrt{1+(1+\frac{1}{\epsilon^*})\delta^2}\,Q(z) \end{bmatrix} \right\|_\infty^2 , \tag{10}$$

where

$$\epsilon^* = \arg\min_{\epsilon>0} \inf_{Q(\cdot)\in H^\infty} \left\| \begin{bmatrix} \sqrt{1+\epsilon}\,(1-P(z)Q(z)) \\ \sqrt{1+(1+\frac{1}{\epsilon})\delta^2}\,Q(z) \end{bmatrix} \right\|_\infty^2 . \tag{11}$$

In particular, when $p_{min} < 2$, we have

$$\epsilon^* = \frac{\delta(p_{min}-\delta)}{1-\delta(p_{min}-\delta)}, \tag{12}$$

and the optimal H^∞ norm becomes

$$\gamma_{opt} = \frac{1}{1+(p_{min}-\delta)^2}, \tag{13}$$

which is the same as that obtained from max-min problem:

$$\sup_{\|\Delta P(\cdot)\|_\infty \le \delta} \inf_{Q(\cdot)\in H^\infty} \left\| \begin{bmatrix} 1-(P(z)+\Delta P(z))\,Q(z) \\ Q(z) \end{bmatrix} \right\|_\infty . \tag{14}$$

Thus the method presented here for solving robust two-block problems always works for the tracking problem. Moreover, it is interesting that for $p_{min} < 2$ the solutions to the min-max problem (6) and the max-min problem (14) coincide. [In general we have min-max $\ge$ max-min.]

References

[1] B. Hassibi and T. Kailath. Tracking with an H^∞ criterion. In *Proceedings of the 36th IEEE Conference on Decision and Control*, San Diego, CA, Dec. 1997.