

An LMI Formulation for the Estimation-Based Approach to the Design of Adaptive Filters¹

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Abstract

We present an LMI formulation for the estimation-based approach to the design of adaptive FIR and IIR filters. Linear Matrix Inequalities (LMI) provide a convenient framework for the synthesis of multi-objective (H_2/H_∞) control problems. Therefore, the H_∞ disturbance attenuation criterion in the estimation-based adaptive algorithm can be easily augmented with the appropriate H_2 performance constraints. The question of internal stability of the overall system is also directly addressed as a byproduct of the Lyapunov-based nature of the LMI formulation. We use an Active Noise Cancellation (ANC) scenario to study the main features of the proposed LMI solution.

1 Introduction

Since the introduction of adaptive algorithms, systematic analysis and synthesis techniques have been of primary interest for the researchers in the field. Inspired by recent developments in the analysis of adaptive algorithms (see [5, 4] and references therein), [6, 7] formulate an estimation-based approach to the synthesis and analysis of the adaptive FIR and IIR filters, and present a Riccati-based solution to the problem. This Riccati-based solution meets an H_∞ disturbance attenuation criterion, and therefore it is conceivable that additional H_2 performance criteria can lead to an improved performance. Researchers (see [3] and references therein) have shown that elementary manipulation of Linear Matrix Inequalities (LMI) can be used to derive less restrictive solutions to the now classical state-space Riccati-based solution to the H_∞ control problem [2]. Further research has developed LMI-based formulations for multi-objective H_2/H_∞ control design (see [8] and references therein). Meanwhile, the availability of fast and efficient tools (such as convex optimization techniques) have made the LMI-based control synthesis practical.

This paper aims at utilizing the well established LMI-based synthesis tools for the systematic design of adaptive filters. The Lyapunov-based nature of the LMI formulation provides a convenient framework in which stability of the overall system can be addressed. It is also straightforward to include other important considerations (such as robustness/performance tradeoffs) in the design process.

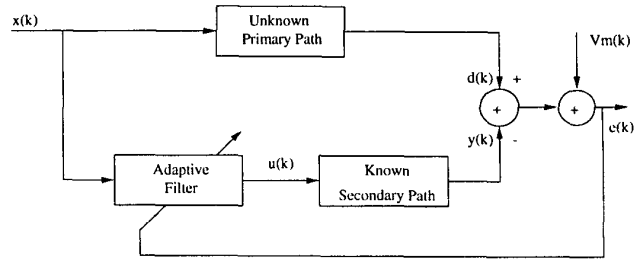


Fig. 1: Block Diagram of an ANC Problem

We use an Active Noise Cancellation (ANC) scenario to examine our LMI-based adaptive filter design technique.

2 Background

We discuss the estimation-based approach for the design of an adaptive filter in the context of the ANC problem of Figure 1. A detailed discussion of the ANC problem and some classical adaptive solutions to the problem can be found in [7]. The objective of ANC is to generate a control signal $u(k)$ such that the output of the secondary path, $y(k)$, is in some measure close enough to the output of the primary path, $d(k)$. For this to occur, the series connection of the adaptive filter (for some optimal setting of its parameters) and the known secondary path must approximate the unknown primary path. This observation directly leads to an *estimation interpretation* of the adaptive control problem [6].

To proceed, we use a state space description, $\begin{bmatrix} A_s(k) & B_s(k) \\ C_s(k) & D_s(k) \end{bmatrix}$, and the state vector $\theta(k)$ to model the secondary path. We also treat the weight vector, $W(k) = [w_0(k) \ w_1(k) \ \dots \ w_N(k)]^T$, as the state vector that captures the dynamics of the adaptive filter. In this paper we pursue the adaptive FIR filter design¹. The state space representation of the system is then

$$\begin{bmatrix} \xi_{k+1} \\ W(k+1) \\ \theta(k+1) \end{bmatrix} = \begin{bmatrix} F_k & 0 \\ \begin{bmatrix} I_{(N+1) \times (N+1)} & 0 \\ B_s(k)h_k^* & A_s(k) \end{bmatrix} \end{bmatrix} \begin{bmatrix} \xi_k \\ W(k) \\ \theta(k) \end{bmatrix} \quad (1)$$

where $h_k = [x(k) \ x(k-1) \ \dots \ x(k-N)]^T$ captures the effect of the reference input $x(\cdot)$. For this system, the derived measured output is

$$m(k) = \begin{bmatrix} H_k \\ D_s(k)h_k^* & C_s(k) \end{bmatrix} \begin{bmatrix} W(k) \\ \theta(k) \end{bmatrix} + v_m(k) \quad (2)$$

where $m(k) = e(k) + y(k)$. Noting the objective of the noise cancellation problem, we choose $s(k) = H_k \xi_k$ as the

¹A parallel formulation can be developed for IIR filters [6].

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quantity to be estimated. Note that $m(\cdot) \in \mathcal{R}^{p \times 1}$, $s(\cdot) \in \mathcal{R}^{q \times 1}$, $\theta(\cdot) \in \mathcal{R}^{r \times 1}$, and $W(\cdot) \in \mathcal{R}^{(N+1) \times 1}$. All matrices are then of appropriate dimensions. We choose

$$\sup_{\mathcal{V}_m, \xi_0} \frac{\sum_{k=0}^M [s(k) - \hat{s}(k|k)]^* [s(k) - \hat{s}(k|k)]}{\xi_0^* \Pi_0^{-1} \xi_0 + \sum_{k=0}^M \mathcal{V}_m^*(k) \mathcal{V}_m(k)} \leq \gamma^2 \quad (3)$$

as the H_∞ criterion by which $\hat{s}(k|k) \triangleq \mathcal{F}(m(0), \dots, m(k))$ (a causal estimate of $s(k)$) is generated. The Riccati-based solution to this problem is presented in [6] and [7]. The next section discusses an LMI-based solution that easily allows for inclusion of H_2 performance constraints.

3 LMI Formulation

Assume the following specific structure for the estimator

$$\hat{\xi}_{k+1} = F_k \hat{\xi}_k + \Gamma_k (m(k) - H_k \hat{\xi}_k) \quad (4)$$

$$\hat{s}_k = L_k \hat{\xi}_k \quad (5)$$

in which Γ_k is the design parameter to be chosen such that the H_∞ criterion (3) is met. Augmenting the system in (1) with (4) and introducing a new variable $\tilde{\xi}_k = \xi_k - \hat{\xi}_k$, the augmented system can be described as

$$\begin{bmatrix} \eta_{k+1} \\ \xi_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi_k & 0 \\ 0 & F_k - \Gamma_k H_k \end{bmatrix} \begin{bmatrix} \eta_k \\ \xi_k \end{bmatrix} + \begin{bmatrix} 0 \\ \Gamma_k \end{bmatrix} \mathcal{V}_m(k) \quad (6)$$

$$Z_k \triangleq s(k) - \hat{s}(k|k) = \begin{bmatrix} 0 & L_k \end{bmatrix} \begin{bmatrix} \xi_k \\ \tilde{\xi}_k \end{bmatrix} \quad (7)$$

The LMI solution for the design of adaptive filters finds a Lyapunov function for the augmented system in (6)-(7) at each step. In other words, at each time step, an infinite horizon problem is solved, and the solution is implemented for the next step. Introducing the quadratic function $V(\eta_k) = \eta_k^T P \eta_k$ (where $P > 0$), it is straightforward to show that for (6) at k , (3) holds if²

$$V(\eta_{k+1}) - V(\eta_k) < \gamma^2 \mathcal{V}_m(k)^T \mathcal{V}_m(k) - Z_k^T Z_k \quad (8)$$

Replacing for Z_k and η_{k+1} from (6)-(7), and after some elementary algebraic manipulations the inequality in (8) can be written as $(\Upsilon_k^T = [\eta_k^T \mathcal{V}_m(k)^T])$

$$\Upsilon_k^T \begin{bmatrix} \Phi_k^T P \Phi_k - P + \Omega_k^T \Omega_k & \Phi_k^T P \Psi_k \\ \Psi_k^T P \Phi_k & \Psi_k^T P \Psi_k - \gamma^2 I \end{bmatrix} \Upsilon_k < 0 \quad (9)$$

Due to the block diagonal structure of Φ_k in (6), the Lyapunov matrix is block diagonal, i.e. $P = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$. Substituting for P in (9)

$$\begin{bmatrix} L_k^T L_k - S + (F_k - \Gamma_k H_k)^T S (F_k - \Gamma_k H_k) & (F_k - \Gamma_k H_k)^T S \Gamma_k \\ \Gamma_k^T S (F_k - \Gamma_k H_k) & \Gamma_k^T S \Gamma_k - \gamma^2 I \end{bmatrix} < 0$$

²Note that (8) is a more conservative inequality compared to the case where the energy of the error in the initial condition of the system is also included

$$F_k^T R F_k - R \leq 0 \quad (10)$$

Since the LMI in (10) can never be strict (F_k has eigenvalues on unit circle) and most SDP-solvers only solve strictly feasible LMI, we first put F_k in its modal form (i.e. $F_k = \Theta_k^{-1} \begin{bmatrix} I & 0 \\ 0 & \Lambda_{A_{s,k}} \end{bmatrix} \Theta_k$ for some Θ_k) to isolate the poles on the unit circle. Straightforward matrix manipulations then lead to the following LMI in S , R_s , $T \triangleq S \Gamma_k$, and γ^2 which can be strictly feasible:

Minimize $\gamma^2 > 0$ subject to

$$\begin{cases} \begin{bmatrix} -S & S(F_k - \Gamma_k H_k) & S \Gamma_k \\ (F_k - \Gamma_k H_k)^T S & L_k^T L_k - S & 0 \\ \Gamma_k^T S & 0 & -\gamma^2 I \end{bmatrix} < 0 \\ \Lambda_{A_{s,k}}^T R_s \Lambda_{A_{s,k}} - R_s < 0 \\ S > 0, \quad R_s > 0 \end{cases} \quad (11)$$

and $R = \Theta_k^T \begin{bmatrix} I & 0 \\ 0 & R_s \end{bmatrix} \Theta_k$. The solution to (11) provides estimator gain, Γ_k , as well as the Lyapunov matrix P which ensures that the quadratic cost $V(\eta_k)$ decreases over time. It is shown in [6, 7] that for a Riccati-based solution to (3) the optimal value of γ is 1. In the absence of H_2 constraints, γ in (11) can be set to 1. This reduces the LMI in (11) to a feasibility problem.

Augmenting the above mentioned H_∞ objective with appropriate H_2 performance constraints is straightforward. One such constraint is the H_2 norm of the transfer function from exogenous disturbance $\mathcal{V}_m(k)$ to the cancellation error Z_k in (7)

$$\|T_{Z\mathcal{V}_m}\|_2^2 = \text{Tr}(\Omega_k W_c \Omega_k^T) \quad (12)$$

where W_c satisfies

$$\Phi_k^T W_c \Phi_k - W_c + \Psi_k \Psi_k^T = 0 \quad (13)$$

Ref. [1] explains in detail how a constraint on the H_2 norm can be translated into an LMI and how a suboptimal solution for the mixed H_2/H_∞ problem can be pursued. We omit the details here due to space limitation.

4 Adaptation Algorithm

The adaptation algorithm is similar to Refs. [6], [7]. Note that the update for $\hat{\xi}_k$ is governed by (4) where the gain vector $\Gamma_k = S^{-1}T$ is the ultimate outcome of the LMI solution to the problem.

1. Start with $\hat{W}(0) = \hat{W}_0$, $\hat{\theta}(0) = \hat{\theta}_0$ as the estimator's best initial guess for the state vector in the approximate model of the primary path. Also assume that $\theta_{actual}(0)$ is the actual initial state of the secondary path. The adaptive algorithm assumes that $\theta_{copy}(0)$ is the initial state for the secondary path. Furthermore, $d(0)$ is the initial output of the primary path.

2. For $0 \leq k \leq M$ (finite horizon):

- Form h_k ,
- Form the control signal $u(k) = h_k^* \hat{W}(k)$,
- Apply the control signal to the secondary path. The actual output and the new state vector obey

$$\theta_{actual}(k+1) = A_s(k) \theta_{actual}(k) + B_s(k) u(k) \quad (14)$$

$$y(k) = C_s(k) \theta_{actual}(k) + D_s(k) u(k) \quad (15)$$

- (d) Propagate the internal copy of the state vector and the output of the secondary path³

$$\theta_{copy}(k+1) = A_s(k)\theta_{copy}(k) + B_s(k)u(k) \quad (16)$$

$$y_{copy}(k) = C_s(k)\theta_{copy}(k) + D_s(k)u(k) \quad (17)$$

- (e) Form the *derived* measurement $m(k) = e(k) + y_{copy}(k)$. Note that $e(k)$ is the error measurement after the control signal $u(k)$ is applied.
- (f) Use the LMI formulation in (11) to find Γ_k (estimator gain) and the subsequent new best estimate of the adaptive filter parameters $\hat{W}(k+1)$, as well as the best estimate of the state of the secondary path. $\hat{\theta}(k+1)$ is needed for the next state update.
- (g) If $k \leq M$, go to (a).

The main difference from what is presented in [7, 6] is step (f) where the estimator gain is calculated. The solution however provides a certificate for the stability of adaptive algorithm by generating the Lyapunov matrix P .

5 Simulation Results

In this section we examine the feasibility of the design procedure, as well as some preliminary performance assessment for the adaptive algorithm of Section 4. Second order systems $P(z) = \frac{10(z-0.3)}{(z+0.6-j0.6)(z+0.6+j0.6)}$ and $S(z) = \frac{10(z-0.3)}{(z+0.01-j0.6)(z+0.01+j0.6)}$ (different resonant frequencies and damping) are used in our simulations. The length of the FIR filter is 4. Simulation results for a multi-tone reference signal ($x(k) = \sum_{i=1}^2 4 \sin(2\pi f_i k \Delta t)$, $f_1 = 10$ and $f_2 = 20$ Hz) are reflected here. The sampling interval is $\Delta t = 0.01$ seconds. For measurement noise, a zero-mean white Gaussian noise with variance 0.05 is used. Fig. 2 shows the time history of the error signal as well as the control signal generated by the FIR filter. The LMI-based adaptive algorithm effectively cancels the disturbance at the output of the primary path, $d(k)$, in approximately 1.0 seconds. Fig. 3 reflects the convergence of the weight vectors in 1.0 seconds.

6 Conclusion

This paper details a new LMI-based synthesis tool for adaptive filter design. The feasibility of this approach is demonstrated in a typical adaptive ANC scenario. One clear benefit is that the framework is suitable for designing adaptive filters in which performance and robustness concerns are systematically addressed. Future work will investigate the robustness/performance tradeoffs in the LMI-based adaptive filter design.

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³ $m(k) = e(k) + y(k)$, where only $e(k)$ is directly measurable. The adaptive algorithm must generate an internal copy of $y(k)$.

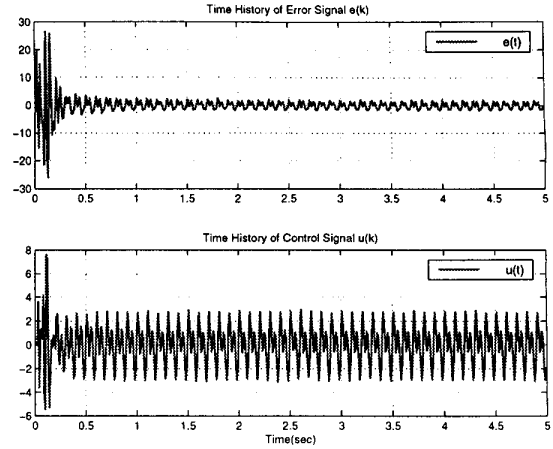


Figure 2: Time History of the Error and Control Signal in the Case of a Multi Tone Input

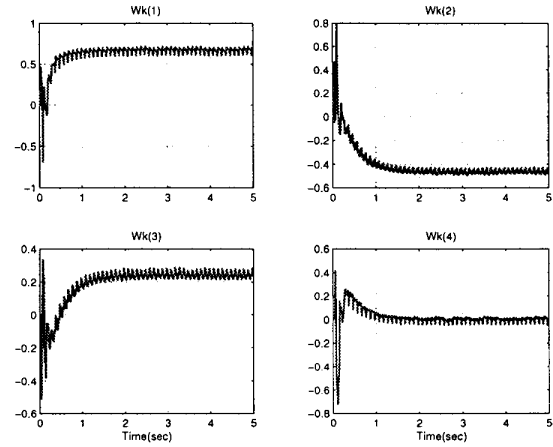


Figure 3: Elements of the Weight Vector in the FIR Filter for the case of a Multi Tone Input

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