MIXED H^2/H^∞ OPTIMAL SIGNAL RECONSTRUCTION IN NOISY FILTER BANKS

Haris Vikalo†, *Babak Hassibi*‡, *and Thomas Kailath*†

†Information Systems Lab, Stanford University, Stanford, CA 94305
‡Bell Laboratories, Lucent Technologies, Murray Hill, NJ 07974

ABSTRACT

We study the design of synthesis filters in noisy filter bank systems using an H^∞ estimation point of view. The H^∞ approach is most promising in situations where the statistical properties of the disturbances (arising from quantization, compression, etc.) in each subband of the filter bank is unknown, or is too difficult to model and analyze. For arbitrary analysis polyphase matrices, standard state-space H^∞ techniques can be employed to obtain numerical solutions. When the synthesis filters are restricted to being FIR, as is often the case in practice, the design can be cast as a finite-dimensional semi-definite program. In this case, we can effectively exploit the inherent non-uniqueness of the H^∞ solution to optimize for an additional average performance and thus obtain mixed H^2/H^∞ optimal FIR synthesis filters.

1. INTRODUCTION

Multirate filter banks systems have been a subject of extensive studies (see [1] and the references therein) and are widely used in many application areas (such as speech and image compression, joint source channel coding, adaptive systems, and others). The design of perfect reconstruction filter banks, capable of exactly replicating the input signal, has received particularly high attention. In most of the research, the subbands of the filter bank system are assumed noise free. Such an M-band filter bank system is illustrated in Figure 1. The analysis filters $H_i(z)$ decompose the input signal into subband components, which are then decimated by a factor of M. The signal is reconstructed by upsampling by a factor of M followed by filtering with synthesis filters $F_i(z)$. Ideally, the synthesis filters are required to exactly reconstruct the delayed version of the input signal. However, the decimated signals in the subbands may be, for example, encoded and transmitted (as in speech comparison applications), or be coded for storage, at which point the signal may be compressed and some information lost. The perfect reconstruction approach studied in the literature, assumes no loss of information in the subbands. However, signal quantization and noise corruption in the subbands, as well as computational roundoff, are always present in practical filter banks systems [2],[3]. Thus, noise in subbands must be carefully considered in systems design.

This work was supported in part by AFOSR under grant F49620-95-1-0525 and NSF under contract ECS-9529325

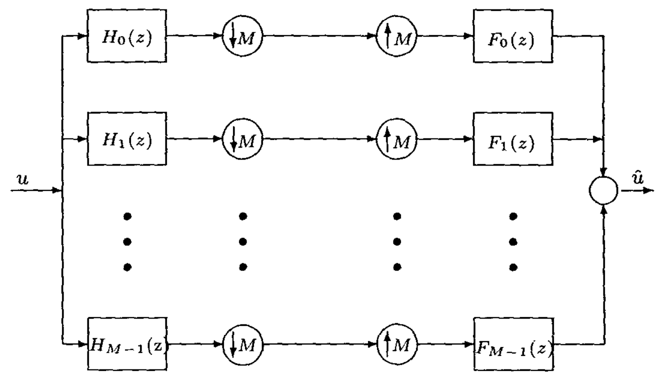

Figure 1: *M-channel filter bank*

In order to deal with noise-corrupted filter bank systems, multirate Kalman synthesis filtering has been recently proposed [4]. The Kalman filtering approaches require a priori knowledge of the (first and second-order) noise statistics. Therefore in applications involving compression, quantization, etc., where the noise statistics are not readily known, the performance of the synthesis filters may be suspect.

H^∞ estimation, on the other hand, requires no statistical assumptions, performs a worst-case design, and is therefore robust with respect to noise uncertainty. The solution to H^∞ optimization problem, however, is highly non-unique (see, e.g., [8]). One way to remove this non-uniqueness is to optimize some other criterion besides the H^∞ feasibility constraint. In this paper, we discuss a particular choice for such a criterion which leads to so-called optimal mixed H^2/H^∞ filter banks. The existence of corrupting noises in the subbands of the filter bank systems is assumed throughout this paper. Analysis filters can be designed for good frequency selectivity (i.e., good coding of the input signal). Then the synthesis filters are designed to minimize the maximum energy gain from the unknown disturbances to the estimation errors, i.e., to minimize the worst-case reconstruction error to disturbance ratio.

2. MODEL DESCRIPTION

To begin our study, we will use a polyphase representation of the filter bank shown in Figure 1. We can represent the

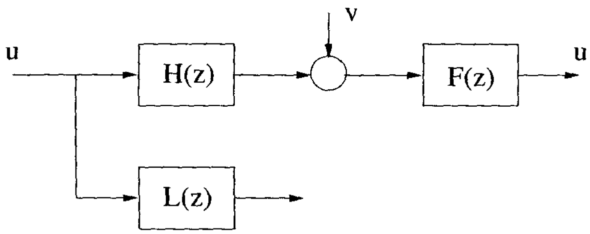

Figure 2: *Vector-matrix equivalent structure*

analysis filter bank in terms of the $M \times M$ polyphase matrix

$$H(z) = \begin{bmatrix} H_{0,0}(z) & H_{0,1}(z) & \cdots & H_{0,M-1}(z) \\ H_{1,0}(z) & H_{1,1}(z) & \cdots & H_{1,M-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ H_{M-1,0}(z) & H_{M-1,1}(z) & \cdots & H_{M-1,M-1}(z) \end{bmatrix}$$

where $H_{k,l}$ is the lth polyphase component of the kth analysis filter. One can find the polyphase analysis matrix $H(z)$ by performing a type-1 polyphase decomposition of the analysis filters as in [1],

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = H(z^M) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}.$$

For the synthesis filter bank, we can define a polyphase matrix $F(z)$ in the similar manner (see, e.g., [1]), and find it by performing a type-2 polyphase decomposition of the synthesis filters,

$$[F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)] = \left[z^{M-1} \quad \cdots \quad z \quad 1\right] F(z^M).$$

Blocking the input and output leads to a so-called vector-matrix equivalent structure in Figure 2. The input signal $\mathbf{u}_i$ (bold symbol denotes a vector) in Figure 2 is of the form

$$\mathbf{u}_j = [u_{iM} \quad u_{iM-1} \quad \ldots \quad u_{iM-M+1}]^T.$$

We are interested in estimating u_{i-m}, the delayed version of the input signal ($m > 0$). The transfer matrix $L(z)$ in Figure 2 can be found as

$$L(z) = z^{-d} \begin{bmatrix} 0_{(M-k)\times k} & I_{(M-k)\times(M-k)} \\ z^{-1} I_{k\times k} & 0_{k\times(M-k)} \end{bmatrix}, \quad m = Md + k,\ k = 0, 1, \ldots, M-1.$$

The system in Figure 2 is the standard model for a general estimation problem, where the goal is to design the causal linear time-invariant estimator $F(z)$ to estimate the input sequence $\{\mathbf{u}_{i-d}\}$ from the observations $\{\mathbf{y}_i\}$. The performance of the estimator is evaluated according to an adopted criterion. For the reasons explained in the introduction, we will focus on H^∞ solution.

The H^∞ optimal solution is, in general, an IIR filter of the same McMillan degree as $[H(z) \quad L(z)]^T$, which could be rather high (see, e.g., [6]). In practice, however, IIR synthesis filters are rarely used in filter bank applications. (One major reason is that finite-precision implementations may lead to limit cycles, or other forms of numerical instability.) Therefore in the remainder of this paper we shall focus on FIR synthesis filters. This has one further advantage: the H^∞ design procedure can be reduced to a *finite* (rather than infinite) dimensional semi-definite program, so that it is possible to effectively optimize the filter weights over criteria in addition to the H^∞ constraint.

3. FIR SYNTHESIS FILTERS: STATE SPACE FORMULATION

The induced transfer matrix mapping the unknown disturbances $\mathbf{u}_i$ and $\sigma^{-1}\mathbf{n}_i$ to the estimation errors is

$$T_F(z) = [L(z) - F(z)H(z) \quad -\sigma F(z)], \qquad (1)$$

where σ^2 represents the noise power. We assume FIR synthesis filters, i.e.,

$$F(z) = F_0 + F_1 z^{-1} + F_2 z^{-2} + \ldots + F_{L-1} z^{-(L-1)}.$$

The state space equations for $T_F(z)$ can be written as

$$\begin{aligned} \mathbf{x}_{i+1} &= A_T \mathbf{x}_i + B_T \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \\ \mathbf{y}_i &= C_T \mathbf{x}_i + D_T \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \end{aligned} \qquad (2)$$

where

$$A_T = \begin{bmatrix} A_H & 0 & 0 \\ 0 & A_L & 0 \\ B_F C_H & 0 & A_F \end{bmatrix},$$

$$B_T = \begin{bmatrix} B_H \\ B_L \\ B_F D_H \end{bmatrix},$$

$$C_T = [-D_F C_H \quad C_L \quad -C_F],$$

$$D_T = -D_F D_H,$$

(A_H, B_H, C_H, D_H) are the matrices in the state space realization of the transfer function $H(z)$, (A_L, B_L, C_L, D_L) are the matrices in the state space realization of the transfer function $L(z)$, and (A_F, B_F, C_F, D_F) are the matrices in the state space realization of the transfer function $F(z)$. It is easy to show that

$$C_F = [F_1 \ldots F_{L-1}], \quad D_F = F_0.$$

Thus the design parameters (that is, $F_0, \ldots F_{L-1}$) appear linearly in C_F and D_F, whereas all other system matrices in (2) are independent of the impulse response of $F(z)$.

We now invoke a standard representation of the H_∞ norm as a convex constraint parametrized over the matrices obtained from the state-space representation.

Theorem 1 *Given matrices A_T and B_T in the state-space realization of $T_F(z)$, the solution to the optimal H^∞ reconstruction problem is given by*

$$\min_{X, C_F, D_F} \gamma$$

subject to

$$\begin{bmatrix} A_T^T X A_T & A_T^T X B_T & C_T^T \\ B_T^T X A_T & B_T^T X B_T - \gamma I & D_T^T \\ C_T & D_T & -\gamma I \end{bmatrix} < 0 \quad (3)$$

$$X > 0$$

Proof: The proof can be found in, e.g., [8]. ∎

Notice that constraint (3) is an LMI (linear matrix inequality) in X, C_T and D_T. This SDP can be solved using efficient algorithms such as the primal-dual method ([7]).

4. MIXED H^2/H^∞ SIGNAL RECONSTRUCTION

As noted in introduction, the solution to the H^∞ estimation problem is highly non-unique. This is due to the fact that the suboptimal H^∞ problem (see, e.g., [6]) is expressed as a feasibility problem, rather than an optimization problem. [Standard implementations of the H^∞ feasibility problem yield the so-called central solution (see, e.g., [5]).] One way to remove this non-uniqueness is to optimize some other criterion besides the H^∞ feasibility constraint. A natural choice for a criterion in a filter bank reconstruction context is to minimize the H^2 norm of the transfer matrix $T_F(z)$,

$$\|T_F\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \text{trace}\left[T_F(e^{j\omega})T_F^*(e^{j\omega})\right]d\omega\right)^{\frac{1}{2}}.$$

By introducing an H^∞ constraint to the H^2 optimization problem, we can exploit the non-uniqueness of the solution to the H^∞ problem in order to improve some other performance aspect of the estimator besides its obvious robustness. This leads to the mixed H^2/H^∞ criterion (see, e.g., [8]), and results in the estimator with the best average performance among all estimators achieving the same optimal γ-level.

Problem 1 (Mixed H^2/H^∞ Signal Reconstruction) *Given $\gamma > 0$, find a causal polyphase synthesis filter $F(z)$ that minimizes the H^2 norm of the transfer function $T_F(z) = [L(z) - F(z)H(z) \quad -\sigma F(z)]$, subject to the H^∞ norm of $T_F(z)$ being less than γ. In other words, find a causal $F(z)$ that satisfies*

$$\begin{aligned} &\min_{F(z)} \|T_F(z)\|_2 \\ &\textit{subject to } \|T_F(z)\|_\infty \le \gamma \end{aligned} \quad (4)$$

Note that, in the frequency domain, both the objective and the constraints are convex. While it is possible to solve the above problem by sampling in the frequency domain, this generally leads to an infinite-dimensional SDP, since we will have an infinite number of constraints (one for each frequency). Therefore, as in the pure H^∞ problem, in order to obtain a finite-dimensional SDP, we seek a way to restate the problem of finding the optimal H^2/H^∞ solution in terms of its state-space representation. To this end, assuming the state-space description of the transfer function $T_F(z)$ in (2), we use the standard representation of the H^2 norm as an LMI constraint to formulate the mixed H^2/H^∞ optimization problem as a SDP in state-space.

Theorem 2 *The mixed H^2/H^∞ signal reconstruction problem (4) is equivalent to the following SDP:*

$$\min_{C_F, D_F} \alpha^2$$

subject to (3),

$$\begin{bmatrix} A_T^T Y A_T - Y & A_T^T Y B_T \\ B_T^T Y A_T & B_T^T Y B_T - I \end{bmatrix} < 0$$

$$\begin{bmatrix} Y & 0 & C_T^T \\ 0 & I & D_T^T \\ C_T & D_T & S \end{bmatrix} > 0 \quad (5)$$

$$Tr(S) - \alpha^2 < 0$$

$$Y > 0.$$

Notice that γ in (3) must be feasible, i.e., we must have

$$\gamma \le \|T_F\|_\infty = \gamma_{opt}.$$

Proof: The proof can be found in [9]. ∎

Moreover, as in the SDP formulation of the pure H^∞ optimization problem, for a given delay and a given analysis filter length, the matrices A_T and B_T are fixed and both (3) and (5) are LMIs in X, Y, S, α^2, and C_T and D_T.

5. SIMULATION RESULTS AND DISCUSSION

In this section, we illustrate the performance of the H^∞ optimal FIR synthesis filters given IIR analysis filters in a 2-band filter bank. As the H^∞ approach does not put any constraints on the choice of the analysis filters, we may choose them arbitrarily. For simplicity, the fifth order Butterworth filters were chosen for the analysis filters. Furthermore, we compare the average performance of the mixed H^2/H^∞ optimal reconstruction filters with the central H^∞ solution as obtained in [5].

Figure 3 shows the largest singular value of the error transfer function, $T_F(e^{j\omega})$ as a function of frequency. Although the area under this curve is not, strictly speaking, the H^2 norm of $T_F(e^{j\omega})$ (since we also need to add the contribution from the second singular value), it is somewhat indicative of the H^2 norm, and hence the average performance of the filters. Figure 3 clearly shows that the H^2-optimal synthesis filter has the smallest area under the curve, which is a result of the fact that, under given stochastic assumptions, the H^2-optimal synthesis filter have the best average performance among all possible causal synthesis filters. On the other hand, the H^∞-optimal synthesis filters yield the error spectra with the smallest peak. Thus the H^∞-optimal synthesis filters (both central and mixed solution) guarantee the best worst-case performance among all causal estimators.

It is also clear from Figure 3 that the H^2-optimal synthesis filter can have poor performance if the disturbance signals have high frequency components, since the value of the error spectrum at these frequencies is large. This is is indicative of the fact that H^2-optimal filters may have poor robustness properties. The H^∞ filters, on the other hand,

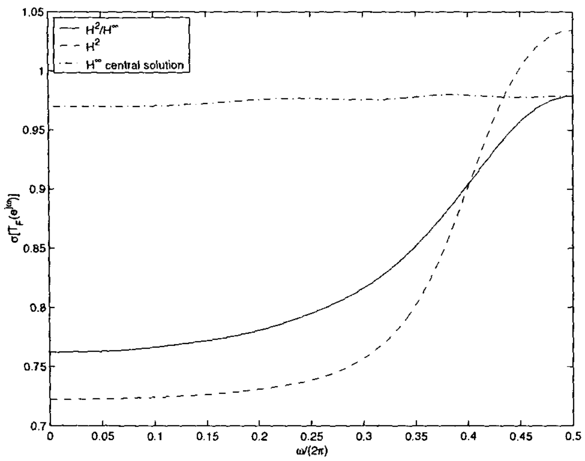

Figure 3: $\bar{\sigma}[T_F(e^{j\omega})]$ vs. ω, $\gamma = \gamma_{opt}$

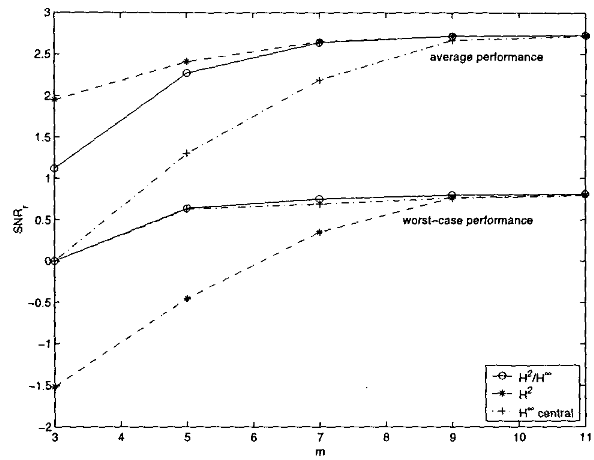

Figure 4: SNR_r vs.m for worst-case input and noise signals.

have inferior average performance. The central H^∞-optimal solution, in particular, has poor average performance, since the curve of $\bar{\sigma}\left[T_F(e^{j\omega})\right]$ is quite flat. The mixed H^2/H^∞ solution, however, by virtue of its very construction, has an area under the $\bar{\sigma}\left[T_F(e^{j\omega})\right]$ curve that is comparable to that of the H^2-optimal solution. Thus it achieves close to optimal average performance, while being robust at the same time.

To compare the performance of the various filters for the signal reconstruction application, we adopt the SNR of the input signal to the reconstruction error ([4])

$$SNR_r = 10\log_{10}\left(\frac{\sum_k u^2(k)}{\sum_k (u(k-m)-\hat{u}(k))^2}\right).$$

Figure 4 compares the average and worst-case performances (in terms of the above reconstruction SNR as a function of the delay, m) for the optimal mixed H^2/H^∞, H^2, and central H^∞ solution. The input signal and noise are modeled as white sequences yielding subband signal-to-noise ratios of 0dB. Thus SNR_r measures the improvement with respect to 0dB obtained from performing reconstruction. As can be seen from Figure 4, the mixed H^2/H^∞ optimal reconstruction filters are on average slightly outperformed by the H^2 optimal reconstruction filters. The average performance of the filters obtained from the central H^∞ solution, however, is significantly worse than that of either the H^2 or mixed H^2/H^∞ one. As mentioned earlier, this is clearly expected from Figure 3. As can be seen, the central H^∞-optimal and mixed H^2/H^∞-optimal filters significantly outperform the H^2-optimal filter, in terms of the worst-case performance.

In summary, the mixed H^2/H^∞-optimal filter appears to achieve the best of both worlds: it has average performance comparable to that of the H^2-optimal solution, while it significantly outperforms this filter in terms of the worst-case performance.

REFERENCES

[1] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[2] R. A. Haddad and N. Uzun, "Modeling, analysis and compensation of quantization effects in M-band subband codecs," *Proc. ICASSP* '93, pp. III-173-176.

[3] P. P. Vaidyanathan, "On coefficient-quantization and computational rounoff effects in lossless multirate filter banks," *IEEE Trans. Signal Processing*, vol. 39, no. 4, pp. 1006-1008, April 1991.

[4] B.-S. Chen, C.-W. Lin, and Y.-L. Chen, "Optimal signal reconstruction in noisy filter banks: multirate Kalman filtering approach," *IEEE Trans. on Signal Processing*, vol. 43, no. 11, November 1995.

[5] T. Chen and B. A. Francis, "Design of multirate filter banks by $\mathcal{H}^\infty$ optimization," *IEEE Trans. on Signal Processing*, vol. 43, no. 12, December 1995.

[6] B. Hassibi, A. H. Sayed, and T. Kailath, *Indefinite-Quadratic Estimation and Control: A Unified Approach to H^2 and H^∞ Theories*, SIAM 1998.

[7] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics, 1994.

[8] K. Zhou, K. Glover, B. Bodenheimer, and J. C. Doyle, "Mixed H^2 and H^∞ performance objectives I and II," *IEEE Trans. on Automatic Control*, vol. 39, no. 8, pp. 1564-1587.

[9] H. A. Hindi, B. Hassibi, S. P. Boyd, "Multiobjective H^2/H^∞-Optimal Control via Finite Dimensional Q-Parametrization and Linear Matrix Inequalities," Proc. ACC, Philadelphia, PA, June 1998.