

On Robust Multiuser Detection

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Abstract

We study the design of multiuser detectors from an H^∞ point of view. The H^∞ approach is most appropriate in the situations where statistical properties of the disturbances are not known or are too hard to model and analyze. The design of the H^∞ optimal FIR multiuser detectors can be efficiently performed using numerical methods. Exploiting the inherent non-uniqueness of the H^∞ solution, we additionally optimize for an average performance thus obtaining mixed H^2/H^∞ optimal multiuser detector. Recursive solutions, allowing for computationally efficient implementation of the decision-feedback detectors, is briefly discussed.

1 Introduction

In recent years, multiuser detection has attracted significant interest due to enhancement of performance that it offers in communication schemes with present multiple-access interference (MAI) (see [1] and references therein). High complexity of the optimal multiuser detectors has led to the development of the various linear multiuser detectors, trading off performance for lower complexity. The so-called decorrelating detector for rejection of multiuser interference was considered in, e.g., [2], [3]. To improve on performance of decorrelating detector at low signal-to-noise ratios (SNR), minimum-mean square error (MMSE) linear multiuser detector was considered in, e.g., [4]-[6]. Further, to deal with multiple-access interference, a recursive MMSE optimal multiuser detector based on Kalman filtering was proposed in [7], [8]. The Kalman filtering requires a priori knowledge of the (first and second-order) noise statistics. However, due to modeling errors and rapid time-variation of the system-parameters that characterize mobile communications, the exact statistics and distribution of the underlying signals are often not known. Further, despite the numerous experimental measurements confirming

that in many physical channels the ambient noise is impulsive, most of the multiuser detection techniques focus on the situation where the noise is additive white Gaussian. Only lately, the problem of robust multiuser detection has been addressed in literature (see, e.g., [9]).

The robust estimation methods (so-called H^∞) design safeguards against worst-case performance, making no assumptions on statistical properties of the signals. This is in contrast to the aforementioned mean-square-error minimizing designs that optimize the average (or expected) performance of the detectors and whose performance heavily depends upon validity of the statistical assumptions made in the design process.

In this paper, we present synthesis procedure for the design of the FIR multiuser detectors based on H^∞ and so-called mixed H^2/H^∞ -optimal design techniques (see, e.g., [10]-[12]). The mixed design allows for trade off between the best average performance and the best worst-case performance. Thus the optimal mixed H^2/H^∞ detectors achieve best average performance not over the set of all detectors but over a restricted set of detectors that achieve certain bound on worst-case performance. The mixed design is allowed for by the fact that the H^∞ solution is highly non-unique. This is due to the nature of the H^∞ problem, which is commonly expressed as a feasibility, rather than an optimization problem. One way to remove the non-uniqueness is to optimize for some criterion other than H^∞ constraint. A natural choice is the H^2 norm, which then leads to the mixed H^2/H^∞ optimal solution. Beside the H^∞ and mixed H^2/H^∞ optimal FIR multiuser detectors, we provide recursive forms which can be implemented via fast algorithms, yielding computationally efficient realization.

2 System model

We consider a multiple-access system with M users, with a total bandwidth LW_0 . Let $s_i(n)$ denote i th user symbol stream, $1 \leq i \leq M$. Each user's symbol stream is modulated by the corresponding signature waveform, $c_i(n)$, and then transmitted. The modulation of the i th user's symbol

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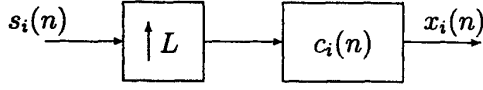


Figure 1. Modulation of the i th user's data stream

stream $s_i(n)$ by a signature waveform $c_i(n)$ can be viewed as upsampling by a factor of L , $L \geq M$, followed by filtering with a signature waveform, as depicted in Figure 1. Output of each such stage is given by

$$x_i(n) = \sum_k s_i(k) c_i(n - k).$$

We assume a slow frequency fading channel where the coherence time of the channel, T_c , is greater than the symbol period of the transmitted signal. Hence, the multiuser channel in Figure 2 is modeled with stationary Rayleigh fading, where $v(n)$ is an additive noise and where (dropping the time dependence index for the simplicity) $p_i(n)$ denotes the channel response corresponding to the i th user. Then the

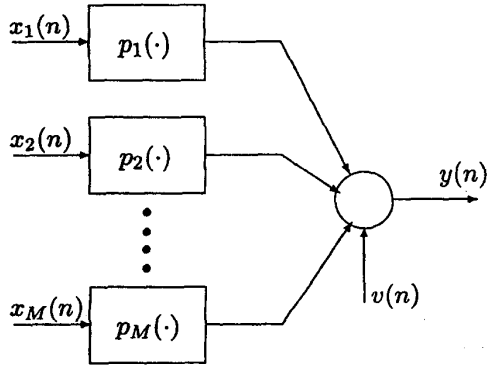


Figure 2. Multiuser fading channel model

received sequence, $y(n)$, is given by

$$y(n) = \sum_i \sum_k p_i(k) x_i(n - k) + v(n).$$

We assume that the channel is known at the receiver. The receiver structure for extracting the symbol stream of the i th user is composed of equalization, followed by demodulation, and is depicted in Figure 3. Assuming discrete-valued, uncoded data symbols, the decision is made upon appropriately thresholding the estimate \hat{s}_i .

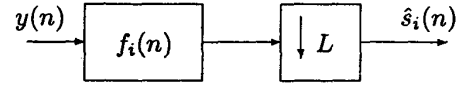


Figure 3. Receiver structure for the i th user

Denote the combination of the signature sequence and the channel response as

$$h_i(n) = \sum_k p_i(k) c_i(n - k).$$

We find it convenient to represent the resulting M -channel bank of filters in polyphase form, that is, find polyphase matrix $\mathbf{H}(z)$. In particular, by performing the Type-2 polyphase decomposition (see, e.g., [13]), we find

$$\mathbf{H}_i(z) = \sum_{k=0}^{M-1} z^k H_{ik}(z^M);$$

where $H_{ik}(z^M)$ is the k th polyphase component of $\mathbf{H}_i(z)$. Similarly, the polyphase representation of the receiver is found by performing the Type-1 polyphase decomposition

$$\mathbf{F}_i(z) = \sum_{k=0}^{M-1} F_{ki}(z^M) z^{-k},$$

where $F_{ki}(z^M)$ is the k th polyphase component of $\mathbf{F}_i(z)$.

3 H^∞ design

With the polyphase representation of the transmitter and receiver as defined in previous section, the design of the multiuser detector can be regarded as a special case of the estimation problem formulation depicted in Figure 4. As

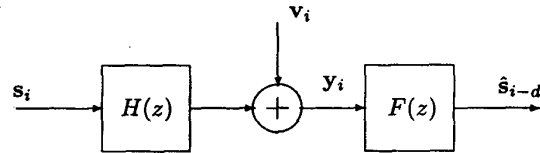


Figure 4. Linear estimation formulation

denoted in the figure, we shall allow for the delay d of the detectors decision. The vectors $\{s_i\}$ and $\{y_i\}$ are obtained by blocking transmitted and received signal, respectively. For example, $\{s_i\}$ in Figure 4 is of the form

$$s_i = [s_1(i) \quad s_2(i) \quad \dots \quad s_M(i)]$$

Let $T_F(z)$ denotes the induced transfer matrix mapping the unknown sequences $\{s_i\}$ and $\sigma^{-1}\{v_i\}$ to the estimation errors, that is,

$$T_F(z) = [z^{-d}I - F(z)H(z) \quad -\sigma F(z)], \quad (1)$$

where σ^2 represents the intensity of the noise.

The goal of H^∞ estimation is to choose a causal $F(z)$ to minimize the H^∞ norm of $T_F(z)$.

Definition 1 (The H^∞ Norm) Let the transfer matrix $T(z)$ maps the input sequence $\{u_i\}$ to the output sequence $\{y_i\}$. Then the H^∞ norm of $T(z)$ is defined as

$$\|T(z)\|_\infty \triangleq \sup_{u \in l^2, u \neq 0} \frac{\|y\|_2}{\|u\|_2} \quad (2)$$

where $\|u\|_2^2 \triangleq \sum_{i=-\infty}^{\infty} u_i^* u_i$ and l^2 denotes the Hilbert space of all square-summable discrete-time signals.

In other words, the H^∞ norm of a stable LTI system is the square-root of its maximum energy gain (more precisely, its l^2 -induced norm). Applied to a multiuser detection problem, we can state the design problem as follows:

Problem 1 (Optimal H^∞ Design) Given a polyphase transmitter matrix $H(z)$, and a delay $d > 0$, find a causal polyphase multiuser detector $F(z)$ that solves for

$$\inf_{F(\cdot)} \sup_{\{s_i, v_i\}} \frac{\sum_i (s_{i-d} - \hat{s}_{i-d})^* (s_{i-d} - \hat{s}_{i-d})}{\sum_i s_i^* s_i + \sigma^{-2} \sum_i v_i^* v_i} \triangleq \gamma_{opt}^2. \quad (3)$$

and find the resulting γ_{opt} .

We should further remark that the frequency domain characterization of the H^∞ norm is given by

$$\|T_F(z)\|_\infty = \sup_{0 \leq \omega \leq 2\pi} \bar{\sigma} [T_F(e^{j\omega})] \quad (4)$$

where $\bar{\sigma}(\cdot)$ denotes the maximum singular value of its argument.

The optimal H^∞ design with a causality constraint in place is hard to solve and, unlike the noncausal case that we just saw, closed-form expression for the causal H^∞ -optimal estimators are generally not available. The common approach is then to relax the optimization condition and solve for a related suboptimal estimation problem. We can state it in the filter bank context as follows:

Problem 2 (Suboptimal H^∞ Design) Given a $\gamma > 0$, find a causal receiver polyphase matrix $F(z)$ that guarantees

$$\|z^{-d}I - F(z)H(z) \quad -\sigma F(z)\|_\infty < \gamma. \quad (5)$$

We notice that it must hold that $\gamma \geq \gamma_{opt}$.

Note that the suboptimal H^∞ problem is expressed as a feasibility problem, rather than an optimization problem. Hence, it turns out to be highly non-unique. This non-uniqueness can be exploited by optimizing for some criterion in addition to the H^∞ constraint.

4 FIR detectors via convex optimization

In this section, we state the H^∞ and mixed H^2/H^∞ optimal FIR detector design problem as a semidefinite program. Let (A_d, B_d, C_d, D_d) be the matrices in the state space realization of the transfer function associated with delay, z^{-d} , and (A_F, B_F, C_F, D_F) be the matrices in the state space realization of the transfer function of the detector, $F(z)$. Further, denote

$$A_T = \begin{bmatrix} A_H & 0 & 0 \\ 0 & A_d & 0 \\ B_F C_H & 0 & A_F \end{bmatrix}, \quad B_T = \begin{bmatrix} B_H \\ B_d \\ B_F D_H \end{bmatrix},$$

$$C_T = [-D_F C_H \quad C_d \quad -C_F], \quad D_T = -D_F D_H. \quad (6)$$

Then we have the following result:

Lemma 1 (Bounded Real) Given any transfer function $T_F(z) = D_T + C_T(zI - A_T)^{-1}B_T$, we have

$$\|T_F\|_\infty < \gamma$$

if and only if the following LMI in X is feasible:

$$\begin{bmatrix} A_T^T X A_T & A_T^T X B_T & C_T^T \\ B_T^T X A_T & B_T^T X B_T - \gamma I & D_T^T \\ C_T & D_T & -\gamma I \end{bmatrix} < 0, \quad X > 0 \quad (7)$$

Notice that for a given transmitter matrix $H(z)$, and a given delay d , once we choose the order of $F(z)$, the matrices A_T and B_T are fixed. Thus, (7) is an LMI (linear matrix inequality) in X , C_T and D_T (and hence in X and the filter weights of $F(z)$, since these appear linearly in C_T and D_T), and we can re-state the suboptimal H^∞ problem as the following semi-definite program (SDP):

Problem 3 (SDP formulation of the H^∞ design problem) Given matrices A_T and B_T in the state-space realization of $T_F(z)$, solve

$$\min_{X, C_T, D_T} \gamma$$

subject to (7).

This SDP can be solved using efficient algorithms such as the primal-dual method ([14], [15]).

As we discussed at the end of the previous section, the solution to the H^∞ design problem is highly non-unique. One way to remove this non-uniqueness is to optimize some other criterion besides the H^∞ feasibility constraint. A natural choice for a criterion in a multiuser detection context is to minimize the H^2 norm of the transfer matrix $T_F(z)$, i.e., $\|T_F\|_2$. We notice that when T_F is a matrix, then

$$\|T_F\|_2 = \text{trace}(T_F T_F^*)^{\frac{1}{2}}.$$

Thus the goal of unconstrained H^2 optimization in the finite horizon case is to minimize the Frobenius norm of the matrix T_F . Introducing an H^∞ constraint to the H^2 optimization problem leads to the mixed H^2/H^∞ criterion.

Problem 4 (Mixed H^2/H^∞ design problem) Given $\gamma > 0$, find a causal polyphase synthesis filter $F(z)$ that minimizes the H^2 norm of the transfer function $T_F(z) = [z^{-d}I - F(z)H(z) - \sigma F(z)]$, subject to the H^∞ norm of $T_F(z)$ being less than γ . In other words, find a causal $F(z)$ that satisfies

$$\begin{aligned} \min_{F(z)} \|T_F(z)\|_2 \\ \text{subject to } \|T_F(z)\|_\infty \leq \gamma \end{aligned} \quad (8)$$

Solution to the mixed H^2/H^∞ problem has the best average performance among all filters achieving the same optimal γ -level. We should also point out that the mixed estimator achieves the optimal H^2 performance whenever the optimal H^2 estimator satisfies the H^∞ bound.

In order to obtain a finite-dimensional SDP, we seek a way to restate the problem of finding the optimal H^2/H^∞ solution in terms of its state-space representation. To this end, assuming the state-space description of the transfer function $T_F(z)$ in (6), we first state a result on the representation of the H^2 norm as an LMI constraint in the following lemma.

Lemma 2 (H^2 Norm bound) Given any transfer function $T_F(z) = D_T + C_T(zI - A_T)^{-1}B_T$, it holds that

$$\|T_F\|_2^2 < \alpha^2$$

if and only if the following LMI in Y and S is feasible:

$$\begin{aligned} \begin{bmatrix} A_T^T Y A_T - Y & A_T^T Y B_T \\ B_T^T Y A_T & B_T^T Y B_T - I \end{bmatrix} < 0 \\ \begin{bmatrix} Y & 0 & C_T^T \\ 0 & I & D_T^T \\ C_T & D_T & S \end{bmatrix} > 0 \\ \text{Tr}(S) - \alpha^2 < 0, \quad Y > 0. \end{aligned} \quad (9)$$

We now use results of Lemma 2 and Lemma 3 to formulate the mixed H^2/H^∞ optimization problem as a SDP in state-space.

Problem 5 (SDP formulation of the H^2/H^∞ problem) The mixed H^2/H^∞ signal reconstruction problem (8) is equivalent to the following SDP:

$$\begin{aligned} \min_{C_T, D_T} \alpha^2 \\ \text{subject to (7), (9)} \end{aligned}$$

Notice that γ in (7) must be feasible, i.e., we must have $\gamma \geq \|T_F\|_\infty = \gamma_{opt}$. Moreover, as in the SDP formulation of the pure H^∞ optimization problem, for a given delay and a given analysis filter length, the matrices A_T and B_T are fixed and both (7) and (9) are LMIs in X, Y, S, α^2 , and C_T and D_T (and hence in the filter weights of $F(z)$, since C_T and D_T are linear in these weights).

5 On recursive implementation

The recursive solution to the estimation problem allows for an efficient implementation of the decision-feedback detectors. To this end, consider the following state-space model

$$\begin{aligned} z_{n+1} &= F z_n + G s_{n+1} \\ y_{n+1} &= P z_n + v_n \end{aligned} \quad (10)$$

where

$$z_n = [s_n \quad \dots \quad s_{n-d+1}]^T, \quad v_n = [v_{nL} \quad \dots \quad v_{nL-L+1}]^T$$

$$F = \begin{bmatrix} 0 & 0 & \dots & 0 \\ I & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad G = \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$P = [p_n \quad p_{n-1} \quad \dots \quad p_{n-d+1}]^T.$$

We are interested in finding filtered estimate of the state, $\hat{z}_{n|n}$. In the H^2 estimation problem, we would like to find $\hat{z}_{n|n}$ such that the expected estimation error energy,

$$E \sum_n (z_n - \hat{z}_{n|n})^* (z_n - \hat{z}_{n|n})$$

is minimized. The solution to this problem is well known and given by the conditional mean of z_n subject to the observation. For $\{s_n\}$ and $\{v_n\}$ independent, zero-mean Gaussian random variables with known variances, $\hat{z}_{n|n}$ is readily found by means of the Kalman filter recursion.

In the H^∞ estimation problem, we would like to find, if possible, set of solutions minimizing

$$\sup_{\{s_n, v_n\}} \frac{\|z - \hat{z}\|^2}{\|s\|^2 + \|v\|^2} < \gamma^2, \quad (11)$$

for some $\gamma > \gamma_{opt}$. We point the reader to [16] for the conditions for the existence of the estimators achieving (11) and parameterization of all possible solutions. [There, a recursive solution to the optimal mixed least-mean-squares/ H^∞ estimation problem is also stated.] Note that the computational complexity of the recursive H^2 and H^∞ (as well as the mixed least-mean-squares/ H^∞ of [16]) optimal estimators is of the same order, cubic in the size of the state vector. When $\hat{z}_{n|n}$ is replaced by a decision obtained upon thresholding, the recursive solution results in a decision-feedback structure.

6 Results and conclusion

We have simulated performance of the two-user system with a model described in Section 2. The spreading sequence of the length 4 is as in [17]. The channel is modeled by frequency selective Rayleigh fading. The FIR detector filters are of the length $T = 6$.

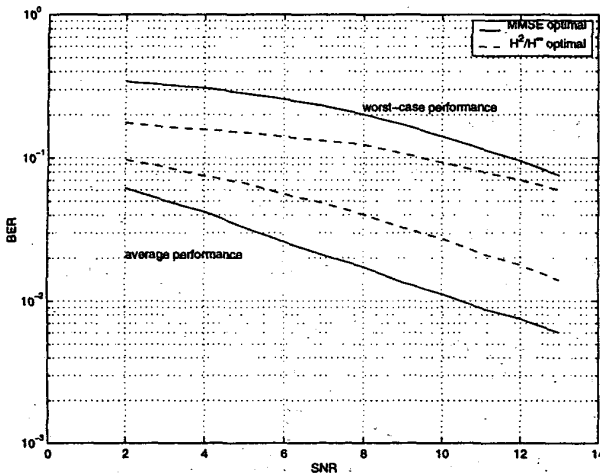


Figure 5. Performance comparison

Figure 5 compares the average and worst-case performances of the MMSE and H^∞ optimal multiuser detectors. As it can be seen from the figure, MMSE optimal detector outperforms H^∞ optimal detector on average. However, under the worst-case scenario (which corresponds to the disturbances that have most of their frequency components at frequencies where $\bar{\sigma}[T_F(e^{j\omega})]$ is large) the H^∞ detectors clearly outperform the MMSE detector.

In summary, we have considered robust H^∞ optimal techniques for the design of the multiuser detectors. Using inherent non-uniqueness of the pure H^∞ design problem, we optimize in addition for an H^2 criterion. The FIR detectors allow for an efficient solution via semidefinite programming. Issues that should further be pursued are regarding computational complexity of the discussed algorithms in the multiuser detection context.

References

- [1] S. Verdu, *Multiuser Detection*, Cambridge University Press, 1998.
- [2] R. Lupas, S. Verdu, "Optimum near-far resistance of linear detectors for code-division multiple-access channels," 1998 IEEE Int. Symp. Information Theory, p. 14, June 1998.
- [3] R. Lupas, S. Verdu, "Near-far resistance of multiuser detectors in asynchronous channels," *IEEE Trans. Communications*, 38:496-508, April 1990.
- [4] Z. Xie, R. Short, and C. Rushforth, "A family of suboptimum detectors for coherent multiuser communications," *IEEE Journal on Selected Areas in Communications*, 8:683-690, May 1990.
- [5] U. Madhow and M. L. Honig, "MMSE interference suppression for direct-sequence spread spectrum CDMA," *IEEE Transactions on Communications*, 42:3178-3188, December 1994.
- [6] H. V. Poor and S. Verdu, "Probability of error in MMSE multiuser detection," *IEEE Transaction on Information Theory*, 43:858-871, May 1997.
- [7] S. H. Isabelle and G. W. Wornell, "Efficient multiuser detectors for intersymbol interference channels," *Signal Processing Advances in Wireless Communications, First IEEE Signal Processing Workshop on*, 1997, pp. 277-280.
- [8] T. J. Lim, L. K. Rasmussen and H. Sugimoto, "An asynchronous multiuser CDMA detector based on the Kalman filter," *IEEE Journal on Selected Areas in Communications*, vol. 16, no. 9, December 1998 (pp. 1711-1722).
- [9] X. Wang and H. V. Poor, "Robust multiuser detection in non-Gaussian channels," *IEEE Transactions on Signal Processing*, vol. 47, no. 2, pp. 289-305, February 1999.
- [10] P. P. Khargonekar and M. A. Rotea, "Mixed H^2/H^∞ filtering," *Proc. IEEE CDC Conference, Tucson, AZ*, December 1992, pp. 2299-2304.
- [11] M. Sznajder, "An exact solution to general SISO mixed H^2/H^∞ problems via convex optimization," *IEEE Transactions on Automatic Control*, vol. 39, no. 3, pp. 2511-2517.
- [12] K. Zhou, K. Glover, B. Bodenheimer and J. C. Doyle, "Mixed H^2 and H^∞ objectives I and II," *IEEE Transactions on Automatic Control*, vol. 39, no. 8, pp. 1564-1587.
- [13] S.-M. Phoong and P. P. Vaidyanathan, "A polyphase approach to time-varying filter banks," *Proc. IEEE ICASSP*, 1996, pp. 1554-1557.
- [14] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics, 1994.
- [15] S.-P. Wu and S. Boyd, *SDPSOL: A Parser/Solver for Semidefinite Programming and Determinant Maximization Problems with Matrix Structure. User's Guide, Beta Version*. Stanford University, 1996.
- [16] B. Hassibi and T. Kailath, "On nonlinear filters for mixed H^2/H^∞ estimation," *Proc. ACC, Albuquerque, NM*, June 1997.
- [17] G. W. Wornell, "Spread-signature CDMA: efficient multiuser communication in the presence of fading," *IEEE Trans. on Inf. Theory*, vol. 41, no. 5, September 1995.