Analysis of Multiple Antenna Wireless Links at Low SNR

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Abstract — **In the low signal-to-noise (SNR) regime when the channel is not known to both transmitter and receiver, we show that the capacity of a multiple antenna Rayleigh fading link is asymptotically quadratic in the SNR for all practical input distributions. This is much less than the known channel case where it exhibits a linear growth in the SNR. Under various signaling constraints we show that mutual information is maximized by using a single transmit antenna. Also sending training symbols offers no advantage at low SNR.**

I. Introduction and Model

Wireless channels with multiple transmit/receive antennas are known to provide a high spectral efficiency both when the channel is known to the receiver, and when the channel is not known to the receiver if the signal-to-noise ratio (SNR) is high. At low SNR, since channel estimates are not reliable, it is usually not reasonable to assume that the channel is known at the receiver.

We use the block fading model of a wireless multiple antenna system proposed by Marzetta and Hochwald in [1], expressing the mutual information between input and output as a function of the model parameter ρ (proportional to the SNR) up to second order.

The M-transmit and N-receive antenna channel is described by a propagation matrix H that is assumed constant for a coherence interval of length T symbols. For each coherence interval the $T \times N$ received matrix X is related to the $T \times M$ transmitted matrix S by

$$X = \sqrt{\frac{\rho}{M}} SH + V, \qquad (1)$$

where H is $M \times N$ and V is a $T \times N$ noise matrix, both comprised of zero-mean and unit variance circularly symmetric complex Gaussian entries. S satisfies the power constraint $\mathrm{EtrSS}^* \leq \mathrm{P}_{\max}$.

II. Main Results

Theorem 1. *Consider the model (1) and let $p(S)$ denote the pdf of S.*

1. **First order result:** *If (i) $\partial p(S)/\partial\rho$ exists at $\rho = 0$ and (ii) $\lim_{\rho\to 0} \rho \mathrm{Etr}(\mathrm{SS}^*)^2 = 0$, the mutual information between the transmitted and received signals S and X for the multiple antenna system (1) is zero to first order in ρ, i.e., $I(X;S) = o(\rho)$.*
2. **Second order result:** *If in addition (i) $\partial^2 p(S)/\partial\rho^2$ exists at $\rho = 0$, (ii) the fourth order moment of S is finite, i.e., $\mathrm{Etr}(\mathrm{SS}^*)^2 < \infty$ and (iii) $\lim_{\rho\to 0} \rho \mathrm{Etr}(\mathrm{SS}^*)^3 = 0$, then the mutual information between S and X up to second order in ρ is given by*

$$I(X;S) = \frac{N\mathrm{tr}[\mathbf{E}(\mathrm{SS}^*)^2 - (\mathbf{E}\mathrm{SS}^*)^2]}{2M^2}\rho^2 + o(\rho^2). \quad (2)$$

The second order part of the theorem is essentially a result in [2] and [3]. However here we require a weaker assumption on the input signals; essentially conditions on the fourth and sixth order moments, rather than an exponentially-decaying input distribution as in [2], or a peak constraint as in [3], both of which render all moments finite.

Note that under any reasonable input distribution (and certainly all practical modulation schemes) the mutual information has no linear term in ρ and so the capacity is *much less* than the known channel case where the low SNR expansion of the well known log det formula has a non-zero first order term.

We impose two practical constraints on S (bounded singular values, constrained fourth order moment) and in each case optimize the mutual information. In the peak constrained case it is possible to obtain a higher capacity by lowering the signal power from its maximum allowed level.

Finally we may apply the result (2) to modulation schemes such as Gaussian modulation and Unitary Space-Time Modulation in which the moments of S are known. For these schemes one transmit antenna is optimal. We also use (2) to show that sending training symbols offers no advantage at low SNR.

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References

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