STATE-SPACE STRUCTURE OF FINITE HORIZON OPTIMAL MIXED H^2/H^∞ FILTER

B. Halder, B. Hassibi and T. Kailath [1]

Information Systems Lab.
Stanford University
Stanford, CA 94305

Abstract. We show that in contrast to the infinite horizon case, the finite horizon optimal mixed H^2/H^∞ estimation problem has a bounded order solution. Moreover, when the underlying systems have state-space models of order n, the optimal finite horizon mixed H^2/H^∞ filter has state-space model of order no greater than $2n+1$. Similar results hold for mixed H^2/H^∞ control, though for brevity we shall only consider estimation problem here.

Keywords. H^2/H^∞ theory, Convex optimization, State-space theory, Robust estimation.

1. INTRODUCTION

In this paper we study the finite horizon mixed H^2/H^∞ estimation problem. In particular, we investigate the structure of the optimal mixed filter. The widely used H^2 theory dominated the classical estimation due to the fact that the optimal H^2 problem has a unique bounded order close-form solution with various desirable optimality properties. However, in the face of model uncertainties and lack of statistical information H^2 methods are not directly applicable and the performance of such estimators is uncertain. Recently, following some pioneering work in robust control theory (Zames, 1981), H^∞ estimation theory has been developed to address such problems. Like the H^2-optimal solution, the socalled *central solution* of the suboptimal H^∞ problem is of bounded order and can be readily computed.

The mixed estimation problem was introduced as an attempt to capture the benefits of both the pure H^2 and H^∞ estimators (Khargonekar and Rotea, 1992; Zhou *et al.*, 1994; Bernstein and Haddad, 1989; Yeh *et al.*, 1992; Sznaier, 1994). However, unlike the H^2 or H^∞ problems with readily computable solutions, there is no known "nice" solution for the mixed problem. On the contrary, it is shown that except for the trivial cases the infinite horizon mixed problem does not have any bounded order solution (Foias *et al.*, 1995; Megretski, 1994). Fortunately, as we shall show, the finite horizon optimal mixed problem always has bounded order solution. The apparent contrast in the structures of the optimal mixed solutions for the finite and infinite horizon cases is due to the difference in the stability requirements for these two cases. For infinite horizon case we always require to have a stable solution. Whereas, for the finite horizon case stability is irrelevant.

In the next two sections we describe the system model and the general finite horizon estimation problem followed by the definition of the mixed problem. To show that optimal mixed solution has bounded order, we first recast the mixed problem into an unconstraint convex optimization problem. Next, using the idea of sub-gradients for convex functions, we derive the optimality conditions for the solution of the optimization problem. The derivation of the optimality conditions exploiting the convex-

[1]

ity of the problem is the key step in this paper. From the optimality conditions we develop an explicit expression for the optimal solution. Finally, using the expression we show that the optimal mixed solution has bounded order. We would like to point out that, though in this paper we only consider the estimation problem, similar results can be derived for the finite horizon mixed control problem.

2. THE DATA MODEL

A general framework for estimation problems is shown in Fig. 1 which subsumes both time variant and time invariant finite horizon estimation problems. The *causal* linear systems $\mathbf{H}$ and $\mathbf{L}$ are assumed to be known and can be represented by finite *lower block triangular* matrices. We further assume that both $\mathbf{H}$ and $\mathbf{L}$ have state space structures of order n, such that we may write

$$\begin{cases} x_{i+1} = A_i x_i + B_i u_i, & \\ y_i = C_i x_i + D_i u_i + v_i, & 0 \le i \le N, \\ z_i = F_i x_i + E_i u_i & \end{cases}$$

where $A_i \in \mathcal{R}^{n\times n}$, B_i, C_i, D_i, F_i and E_i are the known system matrices of compatible dimensions. This implies that $\mathbf{H}$ and $\mathbf{L}$ are simply the impulse response matrices

$$\mathbf{H} = \begin{bmatrix} D_0 & & \\ C_1 B_0 & D_1 & \\ \vdots & \vdots & \ddots \\ C_N A_{N-1}\cdots A_1 B_0 & \cdots & \cdots \; D_N \end{bmatrix},$$

and

$$\mathbf{L} = \begin{bmatrix} E_0 & & \\ F_1 B_0 & E_1 & \\ \vdots & \vdots & \ddots \\ F_N A_{N-1}\cdots A_1 B_0 & \cdots & E_N \end{bmatrix}.$$

In what follows we shall denote sequences such as $\{u_i\}$, $i = 0, \ldots, N$ by u, and simply write $z = \mathbf{L}u$, to denote that $\mathbf{L}$ maps the input sequence $\{u_i\}$ to the output sequence $\{z_i\}$.

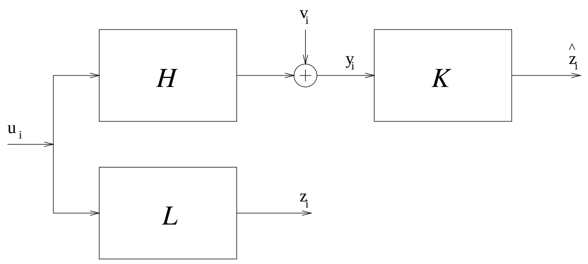

Fig. 1. *A general estimation problem.*

The sequences $\{u_i\}$ and $\{v_i\}$ are assumed to be *unknown.* [$\{u_i\}$ may be considered as a driving disturbance and $\{v_i\}$ as a measurement disturbance. In general, both may include modeling errors resulting from lack of knowledge of the "true" $\mathbf{H}$ and $\mathbf{L}$.] The goal is to design a causal filter, i.e., a lower block triangular matrix of compatible dimension, $\mathbf{K}$ that estimates z_i, the output of $\mathbf{L}$, using the observations $\{y_i, i \le i\}$. The estimates are denoted by $\hat{z}_i$ and the estimation errors by $\tilde{z}_i \triangleq z_i - \hat{z}_i$. From Fig. 1, we can write the induced transfer operator that maps the disturbances $\{u_i\}$ and $\{v_i\}$ to the estimation errors $\{\tilde{z}_i\}$ as

$$\mathbf{T_K} = \begin{bmatrix} \mathbf{L} - \mathbf{KH} & -\mathbf{K} \end{bmatrix}. \tag{1}$$

In other words,

$$\tilde{z} = \mathbf{T_K} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Note that the transfer operator $\mathbf{T_K}$ is an affine function of the matrix $\mathbf{K}$.

3. MIXED H^2/H^∞ ESTIMATION PROBLEM

H^2 and H^∞ techniques represent the two dominant approaches in estimation theory: the classical and the robust approach, respectively. In classical approach, optimality of the *average* (or expected) performance of the estimator is the key issue. Whereas, robust approach safeguards against the *worst-case* performance. However, both the H^2 and H^∞ problems have a common framework and can be stated as: find a causal filter $\mathbf{K}$ such that the H^2 (or the H^∞) norm of the transfer operator is minimized. Since, for matrices the Frobenius norm and the maximum singular value are the H^2 and H^∞ norm, respectively, we can rewrite the H^2 and H^∞ problems as:

$$H^2 \text{ problem:} \quad \min_{\text{causal } \mathbf{K}} \; tr(\mathbf{T_K}\mathbf{T_K}^T)^{1/2},$$

$$H^\infty \text{ problem:} \quad \min_{\text{causal } \mathbf{K}} \; \bar{\sigma}(\mathbf{T_K}),$$

where $\bar{\sigma}$ is the maximum singular value, tr is the trace and T is the transpose. However, the optimal H^∞ problem, as stated above, is hard to solve and often we resort to the following suboptimal problem. Given a scalar γ, find a causal $\mathbf{K}$, if possible, such that $\bar{\sigma}(\mathbf{T_K}) < \gamma$.

Besides the apparently similar forms, the H^2 and H^∞ problem have fundamental differences. Unlike the H^2 estimation problem with unique solution, the solution to the H^∞ estimation problem is highly non-unique. This is quite obvious for the suboptimal problem, but even the *optimal* H^∞ problem has, except for a few special cases, more than one solution. The main motivation of the mixed approach is to effectively exploit this non-uniqueness in order to improve some other aspects of the estimator besides robustness. In particular, the mixed H^2/H^∞ criterion attempts to improve the average performance of the estimator among the class of estimators having the same guaranteed level of robustness.

Unlike the unconstraint H^2 and suboptimal H^∞ problems the socalled "pure" mixed H^2/H^∞ problem of minimizing the H^2 norm, subject to H^∞ norm constraint, has by and large remained a open problem. The above problem is referred to in the literature as the "pure" mixed problem since recently related problems with an auxiliary cost, which replaces the H^2 norm, have been considered (Khargonekar and Rotea, 1992; Sznaier, 1994). One possible reason for studying such various modifications may be that, as several authors have pointed out, the pure mixed problem turns out to be "surprisingly difficult". An indication of this is given in (Foias *et al.*, 1995; Megretski, 1994) where is has been shown that for infinite-horizon LTI problems, the optimal mixed H^2/H^∞ solution has infinite order whenever the H^∞ is not redundant.

Here we are concerned with the pure mixed problem which can be stated as follows.

Problem 1. (Mixed H^2/H^∞ Estimation Problem) *Given a feasible $\gamma > 0$, find a causal estimator* $\mathbf{K}$ *that minimizes the H^2 norm of the transfer operator $\mathbf{T_K} = [\mathbf{L} - \mathbf{KH} \quad -\mathbf{K}]$, subject to the H^∞ norm of $\mathbf{T_K}$ being less than γ. Find a lower block triangular matrix* $\mathbf{K}$*, if possible, that satisfies*

$$\begin{aligned} &\min_{\text{causal } \mathbf{K}} \frac{1}{2} tr(\mathbf{T_K T_K^T}) \\ &\text{subject to} \quad \bar{\sigma}(\mathbf{T_K}) \le \gamma, \end{aligned} \tag{2}$$

where $\bar{\sigma}$ is the maximum singular value, tr is the trace and T is the transpose.

If we denote the H^2-optimal filter by K^2_{opt}, then as mentioned in (Halder *et al.*, 1996), for $\gamma \ge \bar{\sigma}(\mathbf{T}_{\mathbf{K}^2_{opt}})$, K^2_{opt} is also the optimal mixed filter. Hence, we will assume that $\gamma < \bar{\sigma}(\mathbf{T}_{\mathbf{K}^2_{opt}})$. Under this assumption, for any feasible $\gamma > \gamma_{opt}$

$$\bar{\sigma}(\mathbf{T}_{\mathbf{K}^{mix}_{opt}}) = \gamma,$$

where γ_{opt} is the γ corresponding to the optimal H^∞ problem and $\mathbf{K}^{mix}_{opt}$ is the optimal mixed filter. In other words, if the optimal mixed filter exists and differs from the optimal H^2 filter, then the optimal mixed filter attains the H^∞ bound. Hence, for $\gamma_{opt} < \gamma < \bar{\sigma}(\mathbf{T}_{\mathbf{K}^2_{opt}})$, the inequality constraint in (2) is really an equality constraint. Using Lagrange multiplier technique, we recast the mixed problem into an unconstrained convex optimization problem:

$$\min_{\text{causal } \mathbf{K}} \frac{1}{2} tr\{\mathbf{T_K T_K^T}\} + \lambda(\bar{\sigma}(\mathbf{T_K}) - \gamma), \tag{3}$$

where the scalar λ is the Lagrange multiplier. We shall now derive the optimality conditions for the above convex problem.

4. OPTIMALITY CONDITION

Due to the convexity of the problem (3), the first order condition is necessary and sufficient for optimality. However, a straight forward differentiation of the objective function cannot be carried out, since, the maximum singular value is not a differentiable function. We can alleviate this difficulty by using the idea of a sub-gradient of a convex function (Clarke, 1983; Boyd *et al.*, 1991). Unlike gradients, sub-gradients may be non-unique. However, the first order condition requires that only one of the sub-gradients vanish at the optimal point.

As a first step towards the derivation of the optimality condition, we need to find an expression for all the sub-gradients of the objective function. The first term of the objective function $\frac{1}{2} tr\{\mathbf{T_K T_K^T}\}$ is quadratic with respect to the $\mathbf{K}$ and is a differentiable function. There is a unique sub-gradient and it coincides with the usual gradient. We provide an expression for the gradient of the first term in the next result which can be easily verified.

Proposition 1. The gradient of $tr\{\mathbf{T_K T_K^T}\}$ is given by

$$\frac{\partial tr\{\mathbf{T_K T_K^T}\}}{\partial \mathbf{K}} = 2\{\mathbf{K}(\mathbf{I} + \mathbf{HH}^T) - \mathbf{LH}^T\}_{lbt}, \tag{4}$$

where $\{X\}_{lbt}$ denotes the lower block triangular part of X.

Note that the equality implies that the partial derivative of $tr\{\mathbf{T_K T_K^T}\}$ with respect to the lm-th element of $\mathbf{K}$ is equal to the lm-th entry of the matrix on right hand side. Since the indices (l, m) of the matrix $\mathbf{K}$ run over the lower block triangular part, the strictly upper block triangular part of the right hand side is not relevant and the equality makes sense only in the lower block triangular part.

Since, $\mathbf{T_K}$ is affine in $\mathbf{K}$, it is not hard to express all the sub-gradients of the maximum singular value of $\mathbf{T_K}$ (see (Clarke, 1983; Boyd *et al.*, 1991)). The next result provides an expression for all the sub-gradients.

Proposition 2. All the sub-gradients of $\bar{\sigma}(\mathbf{T_K})$ with respect to the l, m element of the matrix $\mathbf{K}$ are given by

$$\frac{\partial \bar{\sigma}(\mathbf{T_K})}{\partial \mathbf{K}} = \{\mathbf{w}\mathbf{q}^T\}_{lbt}. \tag{5}$$

where $\mathbf{w} = (\mathbf{H}\mathbf{p}_1 + \mathbf{p}_2)$ and $\mathbf{q}$ and $\mathbf{p} = \begin{bmatrix} \mathbf{p}_1^T & \mathbf{p}_2^T \end{bmatrix}^T$ are respectively any unit-norm left and right-singular vectors of $\mathbf{T_K}$ corresponding to the maximum singular value.

Proof: First we find an expression for all the sub-gradients of the maximum singular value $X(a)$ with respect to a scalar a. Assuming that X is affine in a, we write the sub-gradients as

$$\frac{\partial \bar{\sigma}(X)}{\partial a} = \mathbf{q}^T \left(\frac{\partial X}{\partial a} \right) \mathbf{p},$$

where $\mathbf{q}$ and $\mathbf{p}$ are any unit norm left and right-singular vectors of $X(a)$, respectively. Now, $\mathbf{T_K}$ is affine in $\mathbf{K}$ (see (1)), hence, all the sub-gradients of $\bar{\sigma}(\mathbf{T_K})$ with respect to the l, m element of the matrix $\mathbf{K}$ is given by

$$\begin{aligned}\frac{\partial \bar{\sigma}(\mathbf{T_K})}{\partial \mathbf{K}_{lm}} &= \mathbf{q}^T \begin{bmatrix} -\mathbf{E}_{lm}\mathbf{H} & -\mathbf{E}_{lm} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}, \\ &= -\mathbf{q}^T \mathbf{E}_{lm}(\mathbf{H}\mathbf{p}_1 + \mathbf{p}_2), \\ &= -\mathbf{q}^T \mathbf{E}_{lm}\mathbf{w},\end{aligned}$$

where $\mathbf{q}$ and $\mathbf{p} = \begin{bmatrix} \mathbf{p}_1^T & \mathbf{p}_2^T \end{bmatrix}^T$ are any unit-norm left and right-singular vectors of $\mathbf{T_K}$ corresponding to the maximum singular value, $\mathbf{E}_{lm}$ is an indicator matrix whose only non-zero at the lm-th position is 1, and $\mathbf{w} = (\mathbf{H}\mathbf{p}_1 + \mathbf{p}_2)$. Rearranging in matrix form, we have

$$\frac{\partial \bar{\sigma}(\mathbf{T_K})}{\partial \mathbf{K}} = \{\mathbf{w}\mathbf{q}^T\}_{lbt}. \qquad \square$$

Using the expressions for the sub-gradients (4) and (5) we write the optimality conditions as follows.

Proposition 3. $\mathbf{K}_{opt}^{mix}$ is a solution to the problem (3), if and only if,

(1) $\bar{\sigma}(\mathbf{T}_{\mathbf{K}_{opt}^{mix}}) = \gamma$.

(2) The matrix $\mathbf{T}_{\mathbf{K}_{opt}^{mix}}$ has a unit-norm right-singular vector $\mathbf{q}$ and a unit-norm left-singular vector $\mathbf{p}$ corresponding to the maximum singular value γ, such that

$$\{\mathbf{K}_{opt}^{mix}(\mathbf{I} + \mathbf{H}\mathbf{H}^T) - \mathbf{L}\mathbf{H}^T - \lambda \mathbf{w}\mathbf{q}^T\}_{lbt} = 0, \tag{6}$$

where $\mathbf{w}$ is already defined in (5).

Note that the vectors $\mathbf{q}$ and $\mathbf{w}$ are non-linear functions of the optimal filter $\mathbf{K}_{opt}^{mix}$ and thus (6) may not be suitable for obtaining a numerical solution for the filter $\mathbf{K}_{opt}^{mix}$. However, relation (6) can be exploited to conclude the finite state-space structure of the optimal mixed filter as shown in the next section.

5. ORDER OF THE OPTIMAL FILTER

In this section, we find a more explicit expression for the optimal filter using the optimality condition (6). Let,

$$\mathbf{I} + \mathbf{H}\mathbf{H}^T = \mathbf{M}\mathbf{M}^T,$$

be the block-LU decomposition of $(\mathbf{I} + \mathbf{H}\mathbf{H}^T)$. Post-multiplying both sides of the equation (6) by the upper block triangular matrix $\mathbf{M}^{-T}$, we get

$$\{\mathbf{K}_{opt}^{mix}\mathbf{M} - (\mathbf{L}\mathbf{H}^T + \lambda \mathbf{w}\mathbf{q}^T)\mathbf{M}^{-T}\}_{lbt} = 0.$$

To arrive at the last equation, we have used the fact that any two matrices with identical lower block triangular parts, when post-multiplied by a upper block triangular matrix, produce identical lower block triangular parts. Rearrangement of the last equation yields,

$$\begin{aligned}\mathbf{K}_{opt}^{mix} &= \{\mathbf{L}\mathbf{H}^T\mathbf{M}^{-T}\}_{lbt}\mathbf{M}^{-1} + \lambda\{\mathbf{w}\mathbf{q}^T\mathbf{M}^{-T}\}_{lbt}\mathbf{M}^{-1}, \\ &= \mathbf{K}_{opt}^2 + \lambda\{\mathbf{w}\bar{\mathbf{q}}^T\}_{lbt}\mathbf{M}^{-1},\end{aligned} \tag{7}$$

where $\bar{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{q}$ and $\mathbf{K}_{opt}^2 = \{\mathbf{L}\mathbf{H}^T\mathbf{M}^{-T}\}_{lbt}\mathbf{M}^{-1}$ is the optimal H^2 filter. Thus, the optimal mixed filter is the sum of two filters, i) the usual H^2 optimal filter which is independent of the H^∞ bound γ, and ii) a second filter which depends on the value of γ and ensures that the mixed filter satisfies the H^∞ constraint. As mentioned earlier, whenever the optimal H^2 filter satisfies the H^∞ constraint, i.e., $\bar{\sigma}(\mathbf{T}_{\mathbf{K}_{opt}^2}) \leq \gamma$, then the Lagrange multiplier is zero ($\lambda = 0$) and the second term vanishes.

In view of (7), we only need to show that the second term in (7), whenever present, has a finite order state-space, since, the optimal H^2 filter always has an n-dimensional state-space realization. To prove the existence of a finite order realization of the second term, we make the following simple but crucial observation.

Proposition 4. The lower block triangular part of any rank one matrix can has an one-dimensional, possibly time varying, state space realization.

As a result, the lower block triangular part of the rank one matrix $\mathbf{w}\bar{\mathbf{q}}^T$ can be realized by a one-dimensional state-space model. Moreover, $\mathbf{M}^{-1}$, the inverse of the casual spectral factor of $(\mathbf{I}+\mathbf{H}\mathbf{H}^T)$, has an n-dimensional state-space structure inherited from $\mathbf{H}$. Again, the product of two matrices with state-space realization of finite orders is guaranteed to have a state-space realization of order no greater than the sum of the individual orders. Thus, the second term in (7) has a state-space realization of order at most $n+1$. Putting these facts together we obtain the next result.

Proposition 5. Assume that $\gamma_{opt} < \gamma < \bar{\sigma}(\mathbf{T}_{\mathbf{K}^2_{opt}})$. Then the optimal mixed filter, has a time varying state space model of order no greater than $2n+1$, where n is the dimensionality of the underlying systems $\mathbf{H}$ and $\mathbf{L}$.

Recall that whenever $\gamma > \bar{\sigma}(\mathbf{T}_{\mathbf{K}^2_{opt}})$ the above proposition is satisfied trivially. This is in contrast to the result obtained for infinite horizon estimation and control where except for trivial cases the mixed optimal solutions have infinite order (Foias *et al.*, 1995; Megretski, 1994). This is due to the fact that stability of the finite horizon solutions is irrelevant, whereas in the infinite horizon scenario stability is a nontrivial requirement.

Besides the order of the optimal filter there are two more interesting facts we would like to stress. First, even if the underlying systems $\mathbf{H}$ and $\mathbf{L}$ are time invariant the optimal filter is time varying whenever the second term involving the singular vectors is present. Moreover, the presence of the second term makes it impossible for the optimal filter to have a recursive solution. In other words, the mixed optimal filter obtained for block length of N is, in general, completely different from the one obtained for block length $N+1$. To see this, note that the addition of a single block-row and block-column to any matrix, in general, changes all the elements of the singular vectors.

6. CONCLUSION

In this paper we studied the state space structure of the optimal mixed H^2/H^∞ filter for finite horizon discrete time systems. The key step is to to derive the optimality condition exploiting the convexity of the problem. The optimality condition shows that the optimal filter can be written as a sum of two filters, both of which have finite order, possibly time varying, state-space structure. This is in contrast to the results obtained for infinite horizon mixed H^2/H^∞ estimation and control where except for trivial cases the optimal mixed solutions are not of bounded order. The finite order result is encouraging and may help to develop the actual methods of computing the solution. However, we must point out that the mixed filter does not yield a recursive solution in the sense that the solution depends on the chosen horizon, N. Finally, we should remark here, although not addressed here, smiler results hold for the closely related mixed H^2/H^∞ control problem.

7. REFERENCES

G. Zames (1981). Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses, In *Proc. IEEE Trans. on Automatic Control*, AC-26, pp. 301-320.

P. P. Khargonekar and M. A. Rotea (1992). Mixed H^2/H^∞ filtering. In *Proc. IEEE Conference on Decision and Control*, Tucson, AZ, Dec. 1992, pp. 2299-304.

K. Zhou, K. Glover, B. Bodenheimer and J. C. Doyle (1994). Mixed H^2 and H^∞ performance objectives I and II. *IEEE Transactions on Automatic Control*, vol.39, no.8, pp. 1564-1587.

D. S. Bernstein and W. M. Haddad (1989). LQG control with an H^∞ performance bound: A Riccati equation approach. *IEEE Transactions on Automatic Control*, vol.34, no.3, pp. 293-305.

H. H. Yeh, S. S. Banda and B. C. Chang (1992). Necessary and sufficient conditions for mixed H^2 and H^∞ optimal control. *IEEE Transactions on Automatic Control*, vol.37, no.3, pp. 355-358.

M. Sznaier (1994). An exact solution to general SISO mixed H^2/H^∞ problems via convex optimization. *IEEE Transactions on Automatic Control*, vol.39, no.3, pp. 2511-2517.

C. Foias, A. E. Frazho and W. S. Li (1995) On H^2 minimization for the Caratheodory-Schur interpolation problem, *Integral Equations and Operator Theory*, vol.21, no.1, pp. 24-32.

A. Megretski (1994) On the Order of Optimal Controllers in the Mixed H^2/H^∞ Control, In *Proc. CDC*, pages 3173–3174, Dec. 1994.

S. Boyd, and C. Barratt(1991). *Linear Control Design: Limits of Performance.* Prentice Hall.

F. H. Clarke(1983). *Optimization and Nonsmooth Analysis.* John Wiley & Sons.

B. Halder, B. Hassibi, and T. Kailath (1996) Mixed H^2/H^∞ Estimation: Primary Analytic Charecterization and a Numerical Solution, In *Proc. IFAC*, pages 37–42, vol. J, June 1996.