Linearly Combined Suboptimal Mixed H_2/H_∞ Controllers*

B. Halder, B. Hassibi and T. Kailath †
Information Systems Lab.
Stanford University, CA 94305

March 5, 1997

Abstract

In this paper we propose some new solutions to the full information mixed H_2/H_∞ state feedback problem: the H_2 norm of the closed-loop system is minimized over a class of linearly combined state feedback laws that satisfy the specified H_∞ bound. Such mixed linearly combined controllers yield smaller closed-loop H_2 norms than those obtained by using the central solutions of the H_∞ problem. Moreover, the mixed controllers achieve the optimal H_2 performance whenever the optimal H_2 controller satisfies the H_∞ bound.

1 Introduction

Unlike the H_2 optimal control problem, the solution to the H_∞ suboptimal control problem is highly non-unique [6]. This is not surprising, since the suboptimal H_∞ problem is posed as a feasibility problem rather than an optimization problem. However, except for some special cases, even for the optimal H_∞ problem there are many solutions. One way to remove this non-uniqueness is to optimize some other desirable criterion besides the H_∞ constraint. The H_2 norm of the closed-loop system is one such criterion that can be considered along with the H_∞ constraint and the resulting optimization problem is referred to as the pure mixed H_2/H_∞ problem. As it turns out, the the pure mixed H_2/H_∞ problem is surprisingly hard and has by and large remained an open problem. However, various modifications to the pure mixed H_2/H_∞ problem have been suggested, e.g., the mixed H_2/H_∞ problem suggested by Bernstein and Haddad [1], and its dual proposed by Doyle and Zhou et al. [2, 3]. Bernstein and Haddad proposed to minimize an auxiliary cost that is an upper bound on the H_2 norm of the closed-loop system. In [4] Khargonekar and Rotea developed a convex formulation for the state feedback case using the same auxiliary cost, and this led to a

*This work was supported in part by DARPA through the Department of Air Force under contract F49620-95-1-0525-P00001 and by the Joint Service Electronics Program at Stanford under contract DAAH04-94-G-0058-P00003.
†Email addresses (in order): `bjt@isl.stanford.edu`, `hassibi@ isl.stanford.edu`, and `tk@isl.stanford.edu`.

computationally efficient solution for the mixed problem. However, as the numerical results show there are two drawbacks to this auxiliary cost approach: i) the true H_2 norm of to the optimal modified mixed H_2/H_∞ solution can even be worse than that of the central solution and ii) the modified mixed H_2/H_∞ solution fails to achieve the optimal H_2 performance when the specified H_∞ norm bound is larger than the H_∞ norm of the H_2 optimal solution [5]. As a result, the minimization of the upper bound may not be an effective way to reduce the true H_2 norm of the closed-loop system. This motivated us to reconsider the original mixed problem with the true H_2 norm.

However, due to the inherent difficulty of the optimal mixed problem, in this paper we consider only a suboptimal approach. The main objective is to obtain an easily computable solution that yields: i) better H_2 performance than the central solution and ii) recovers the optimal performance whenever possible. We propose a state feedback construction based on the optimal H_2 solution and the central solution to the H_∞ problem. The construction hinges on a linear combination of certain dual Riccati variables; hence the name linearly combined feedback. The proposed linearly combined mixed H_2/H_∞ controller achieves the two goals mentioned, though of course it is a suboptimal solution.

2 The Problem

Consider the following linear time invariant model:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w_1 + B_2 w_2(t) + Bu(t), \quad x(0) = x_0, && (1) \\ z(t) &= \begin{bmatrix} Cx(t) \\ Du(t) \end{bmatrix}, \quad D^T D = I, && (2) \end{aligned}$$

where all the matrices A, B, B_1, B_2, C and D are real and constant. The use of (2) is preferred over the usual form to avoid cross-terms. The assumption $D^T D = I$ is equivalent to the full column rank assumption and differs only by a scaling. We assume that (A, B) is controllable and (C, A) is observable. For the given model the pure H_2/H_∞ static state feedback problem can be stated as:

Given an achievable H_∞ bound γ find a internally stabilizing static state feedback $u(t) = Kx(t)$ which satisfies

$$\min_K \|\mathbf{T}_{zw_2}\|_2, \quad \textit{subject to} \quad \|\mathbf{T}_{zw_1}\|_\infty \le \gamma, \tag{3}$$

where $\mathbf{T}_{zw_i}$, $i = 1, 2$, denote the closed-loop map from w_i to z and $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the H_2 and H_∞ norms, respectively.

As mentioned earlier, finding the optimal solution to the above highly nonlinear (and non-convex) multi-dimensional optimization problem is difficult. However, an easily computable sub-optimal solution can be obtained if we restrict the feedback matrix K to some special class. The proposed linearly combined state feedback is one such class in which all possible feedback matrices

K are constructed from a suitable combination of the H_2 optimal solution and the central solution. Moreover, the class of linearly combined state feedbacks can be parameterized by a scalar. We exploit the scalar parameterization to simplify the multi-dimensional optimization problem (3) to a scalar optimization problem that can be easily solved. However, as expected, optimality is lost by restricting the search over the class of linearly combined feedback. The the optimal H_2 solution and the central solution to the H_∞ problem play a vital role in our approach and we review them in the next section along with some other relevant concepts.

3 Preliminaries

The H_2 optimal state feedback is given by $K_2 = -B^T P_2$, where P_2 is the solution of the following Riccati equation:

$$P_2 A + A^T P_2 - P_2 B B^T P_2 + C^T C = 0. \tag{4}$$

Note that under the assumption of controllability and observability the matrix P_2 is guaranteed to be positive definite.

Similarly, the central solution to the sub-optimal γ-level H_∞ problem is given by $K_c = -B^T P_c$, where the positive semidefinite matrix P_c is the solution of the following Riccati equation:

$$P_c A + A^T P_c - P_c(BB^T P_c - \gamma^{-2} B_1 B_1^T P_c) + C^T C = 0, \tag{5}$$

such that $A - (BB^T - \gamma^{-2} B_1 B_1^T) P_c$ is asymptotically stable. Note that the existence of such P_c is also the necessary and sufficient condition for existence of a state feedback K that yields $\|\mathbf{T}_{zw}\|_\infty \leq \gamma$. We assume the existence of P_c, i.e., the given γ-level is achievable. Comparing (5) with (4) we see that $P_c \geq P_2$ and, hence, P_c is also positive definite if it exists.

From the above results, we observe that both K_2 and K_c are of the form $K = -B^T P$, known as the LQR form, for some positive definite matrix P. For the LQR class of feedback matrices K the closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= (A - BB^T P)x(t) + B_1 w_1(t) + B w_2(t), \;\; x(0) = x_0, \\ z(t) &= \begin{bmatrix} C \\ -DB^T P \end{bmatrix} x(t). \end{aligned}$$

We say the feedback $K = -B^T P$ is admissible if the corresponding closed-loop system matrix $(A - BB^T P)$ is stable. Note that both K_2 and K_c are admissible. Now, from the bounded real lemma [6] we know that for an admissible $K = -B^T P$, the closed-loop map $\|\mathbf{T}_{zw_1}\|$ has H_∞ norm $\leq \gamma$ if and only if there exists an $X > 0$ such that

$$X(A - BB^T P) + (A - BB^T P)^T X + \gamma^{-2} X B_1 B_1^T X + PBB^T P + C^T C \leq 0.$$

The existence of such X is known to be equivalent to the condition that the Hamiltonian matrix

$$M(P) = \begin{bmatrix} A - BB^T P & \gamma^{-2} B_1 B_1^T \\ -PBB^T P - C^T C & -(A - BB^T P)^T \end{bmatrix}. \tag{6}$$

has no imaginary eigenvalues.

Again, for an admissible feedback K, the H_2 norm of the closed-loop system $\mathbf{T}_{zw_2}$ is $\|\mathbf{T}_{zw_2}\|_2 = \{Tr(B_2 Y B_2^T)\}^{1/2}$ where Y is the solution of the Lyapunov equation

$$Y(A - BB^T P) + (A + BB^T P)^T Y + PBB^T P + C^T C = 0. \tag{7}$$

4 Linearly Combined State Feedback

We construct the linearly combined feedback $K = -B^T P$ where P is computed using the matrices P_2 and P_c as follows. Since, both P_2 and P_c are positive definite, $Z_2 \triangleq P_2^{-1}$ and $Z_c \triangleq P_c^{-1}$ exist and let

$$Z_\alpha = \alpha Z_c + (1 - \alpha) Z_2, \quad \alpha \in [0, 1], \tag{8}$$

be the linear (convex) combination. The linearly combined feedback $K_\alpha = -B^T P_\alpha$, where $P_\alpha = Z_\alpha^{-1}$. We would like to stress that here we are taking the linear combination of the dual variables Z_2 and Z_c and, hence, the feedback K_α is different form a direct linear combination of K_2 and K_c. Note that the set of linearly combined feedback K_α is parameterized by the scalar α, and that K_2 and K_c are the two extreme points corresponding to $\alpha = 0$ and $\alpha = 1$, respectively. We now show certain desirable properties of this linearly combined feedback K_α.

5 Properties of the Linearly Combined Feedback

Lemma 1 *The linearly combined feedback K_α is admissible for $\alpha \in [0, 1]$. i.e., the closed-loop system matrix $A - BB^T P_\alpha$ is stable.*

Proof: Let $Z > 0$ such that

$$AZ + ZA^T + ZC^T CZ - BB^T \leq 0. \tag{9}$$

Rewriting the last inequality we get

$$(A - BB^T Z^{-1})Z + Z(A - BB^T Z^{-1})^T + ZC^T CZ + BB^T \leq 0.$$

Since, (A, B) is controllable and $ZC^T CZ \geq 0$, the matrix $A - BB^T Z^{-1}$ is stable. In other words, $A - BB^T Z^{-1}$ is stable for all $Z > 0$ that satisfies (9). Note that (9) can be expressed as an LMI in Z as follows:

$$\begin{bmatrix} AZ + ZA^T - BB^T & ZC^T \\ CZ & -I \end{bmatrix} \leq 0.$$

Hence, the set of $Z > 0$ that satisfies (9) is convex. Moreover, from (4) and (5) we observe that both Z_2 and Z_c belongs to the set and so does any convex combination Z_α as given in (8). $\square$

Hence, for the linearly combined feedback K_α the multidimensional optimization problem (3) reduces to finding the $\alpha \in [0,1]$ which minimizes $\|\mathbf{T}_{zw_2}\|_2$ and satisfies the H_∞ constraint.

Lemma 2 *The matrix P_α is an upper bound for Y_α, i.e., $P_\alpha - Y_\alpha \geq 0$ for $\alpha \in [0,1]$, where*

$$Y_\alpha(A - BB^T P_\alpha) + (A + BB^T P_\alpha)^T Y_\alpha + P_\alpha BB^T P_\alpha + C^T C = 0. \tag{10}$$

Proof: Rewriting (4) and (5) in terms of the dual variables Z_2 and Z_c, respectively, we get

$$\begin{aligned} AZ_2 + Z_2 A^T + Z_2 C^T C Z_2 - BB^T &= 0, & (11)\\ AZ_c + Z_c A^T + Z_c C^T C Z_c - BB^T + \gamma^{-2} B_1 B_1^T &= 0. & (12) \end{aligned}$$

Taking the convex combination of (11) and (12), we get

$$AZ_\alpha + Z_\alpha A^T + (1-\alpha) Z_2 C^T C Z_2 + \alpha Z_2 C^T C Z_2 - BB^T + \alpha\gamma^{-2} B_1 B_1^T = 0,$$

where Z_α is as defined in (8). Therefore, $P_\alpha (= Z_\alpha^{-1})$ satisfies the following Riccati equation

$$P_\alpha A + A^T P_\alpha - P_\alpha BB^T P_\alpha + \alpha\gamma^{-2} P_\alpha B_1 B_1^T P_\alpha + C^T C + \Delta = 0, \tag{13}$$

where

$$\begin{aligned} \Delta &= P_\alpha (1-\alpha) Z_2 C^T C Z_2 P_\alpha + \alpha P_\alpha Z_2 C^T C Z_2 P_\alpha - C^T C, \\ &= \alpha(1-\alpha) P_\alpha (Z_2 - Z_c) C^T C (Z_2 - Z_c) P_\alpha, \end{aligned} \tag{14}$$

and hence, $\Delta \geq 0$ for $\alpha \in [0,1]$.

Now subtracting (10) from (13) we get

$$(P_\alpha - Y_\alpha)(A - BB^T P_\alpha) + (A - BB^T P_\alpha)^T (P_\alpha - Y_\alpha) - \alpha\gamma^{-2} P_\alpha B_1 B_1^T P_\alpha + \Delta = 0.$$

This is a Lyapunov equation in $(P_\alpha - Y_\alpha)$ and since A_p is stable and $(\alpha\gamma^{-2} P_\alpha B_1 B_1^T P_\alpha + \Delta \geq 0)$, we conclude that $P_\alpha - Y_\alpha \geq 0$. $\square$

Now, the partial derivative of P_α

$$\dot{P}_\alpha \triangleq \frac{\partial P_\alpha}{\partial \alpha} = P_\alpha (Z_2 - Z_c) P_\alpha,$$

is a positive semi-definite, hence, the upper bound P_α is a monotonically increasing function of α. In view of this fact, we replace the minimization of $Tr(B_2 Y_\alpha B_2^T)$ over α by

$$\inf_{\alpha \in [0,1]} \alpha, \quad \text{subject to} \quad M(P_\alpha) \text{ has no imaginary eigenvalue.} \tag{15}$$

Minimization of $Tr(B_2 Y_\alpha B_2^T)$ over α and (15) will be equivalent if $Tr(B_2 Y_\alpha B_2^T)$ itself is a monotonically increasing function of α. In fact, numerical experiments suggest such result and we have the following conjecture.

Conjecture: *The matrix $\dot{Y}_\alpha$, the partial derivative of Y_α with respect to α, is a positive semidefinite matrix for $\alpha \in [0,1]$.*

Note that $\dot{Y}_\alpha$ satisfies the following Lyapunov equation

$$\dot{Y}_\alpha(A - BB^T P_\alpha) + (A - BB^T P_\alpha)^T \dot{Y}_\alpha + (P_\alpha - Y_\alpha)BB^T \dot{P}_\alpha + \dot{P}_\alpha BB^T (P_\alpha - Y_\alpha) = 0.$$

Unfortunately, the matrix $(P_\alpha - Y_\alpha)BB^T \dot{P}_\alpha + \dot{P}_\alpha BB^T (P_\alpha - Y_\alpha)$ is not necessarily a positive semidefinite matrix. However, the mixed inertia of the above matrix does not exclude the possibility that $\dot{Y}_\alpha$ is positive semidefinite.

The minimum $\alpha \in [0,1]$ which satisfies the H_∞ bound can be efficiently computed using bisection method as described in the next section. Note that, whenever γ is larger than the γ achieved by the H_2 optimal controller, $\alpha = 0$ is the minimum and the mixed controller is identical to the optimal H_2 controller.

6 Algorithm Based on Bisection method

We propose the following bisection algorithm to solve the minimization problem (15).

Step 1: *Given a feasible γ, find Z_2 and Z_c by solving (11) and (12), respectively.*

Step 2: *Find optimal α using bisection method:*

Set $\alpha_u = 1$ and $\alpha_l = 0$;
repeat{$\alpha = (\alpha_u + \alpha_l)$; compute P_α;
form $M(P_\alpha)$ using (6),
if $M(P_\alpha)$ has imaginary eigenvalues, set $\alpha_l = \alpha$,
else set $\alpha_u = \alpha$ }
until $\alpha_u - \alpha_l \leq tolerance$.

Step 3: *Set $K = -B^T P_\alpha$.*

7 Numerical Result

In this section we consider a numerical example to demonstrate the performance of the suboptimal mixed controller relative to the central controller. Following are the system matrices for SISO system considered this example.

$$A = \begin{bmatrix} -8.3405 & -9.2471 & 2.2094 & -2.8908 \\ 8.4203 & 9.4584 & -3.0424 & 3.6046 \\ 3.9078 & 5.3124 & -1.9280 & 5.4118 \\ -0.2490 & 0.2069 & 1.3889 & 0.8869 \end{bmatrix},$$

$$B = \begin{bmatrix} 8.6825 \\ -7.4325 \\ 7.1205 \\ 2.0532 \end{bmatrix}, B_1 = \begin{bmatrix} 0.9085 \\ 0.4571 \\ -0.8233 \\ -1.2795 \end{bmatrix}, B_2 = \begin{bmatrix} 0.6798 \\ 1.8148 \\ -0.0179 \\ -0.3708 \end{bmatrix}, C = \begin{bmatrix} 0.2017 \\ 0.2534 \\ -0.1234 \\ 0.4180 \end{bmatrix}^T, D = 1. \quad (16)$$

For this system, the optimal H_∞ norm of $\mathbf{T}_{zw_1}$ is approximately 3.629 and the H_2 norm of $\mathbf{T}_{zw_2}$ for the central controller is 118.1. On the hand, the optimal H_2 controller yields $\|\mathbf{T}_{zw_1}\|_\infty = 6.3292 (\triangleq \gamma_2)$ and $\|\mathbf{T}_{zw_2}\|_2 = 61.3543$. In Figure (1) we plot the H_2 norm of $\mathbf{T}_{zw_2}$ achieved by the central controller and the linearly combined mixed controller as a function of the parameter γ. The plot shows that we always get some reduction in $\|\mathbf{T}_{zw_2}\|_2$ by using the proposed suboptimal mixed controller. Moreover, the mixed controller achieves the optimal H_2 performance for $\gamma \geq \gamma_2$, as expected.

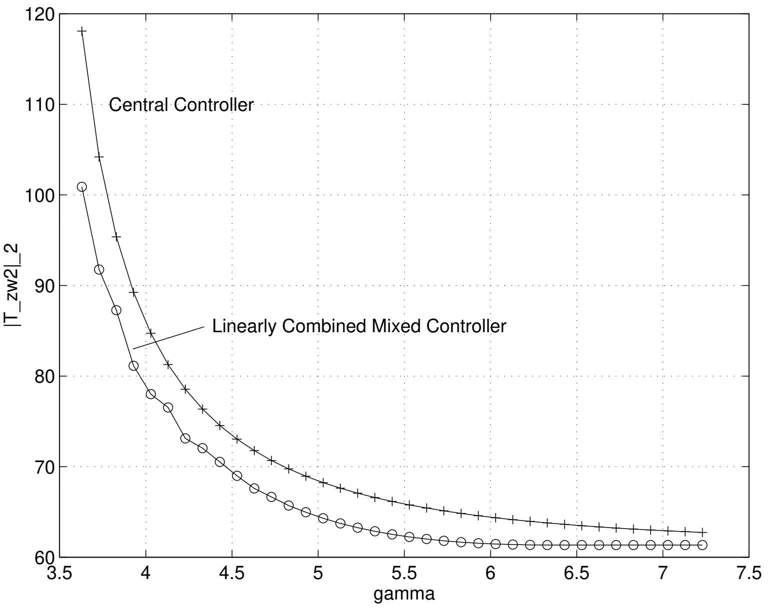

Figure 1: $\|\mathbf{T}_{zw_2}\|_2$ achieved by the central controller ('+') and the linearly combined mixed controller ('o') as a function of γ for the system described in (16). The optimal $\gamma = 3.629$ and the $\|\mathbf{T}_{zw_1}\|_\infty$ for the H_2 optimal controller is $\gamma_2 = 6.3292$.

8 Conclusions

In this paper we considered the full information pure mixed H_2/H_∞ static feedback control problem and proposed a easily computable suboptimal solution. The suboptimal mixed controller yields a smaller closed-loop H_2 norm than that of the H_∞ central solution. Moreover, the suboptimal mixed controller recovers the optimal H_2 performance whenever specified γ is larger than the γ achieved by the H_2 optimal controller. The suboptimal solution is obtained by restricting the controllers to a class of linearly combined controllers that are constructed using the solutions of the H_2 Riccati equation and the H_∞ Riccati equation. Since, the set of linearly combined controllers can be parameterized by a scalar α , the associated multi-dimensional minimization problem over this class reduces to a scalar minimization problem. Exploiting the properties of the linearly combined controllers, we further simplify the problem to finding the minimum α, that satisfies the H_∞ constraint. A bisection method, similar to the method used to compute the H_∞ norm, is then applied to find the minimum α.

References

[1] D. S. Bernstein and W. M. Haddad (1989), LQG control with an H_∞ performance bound: A Riccati equation approach, *IEEE Transactions on Automatic Control*, vol.34, no.3, pp. 293-305.

[2] K. Zhou, K. Glover, B. Bodenheimer and J. C. Doyle (1994), Mixed H_2 and H_∞ performance objectives I and II, *IEEE Transactions on Automatic Control*, vol.39, no.8, pp. 1564-1587.

[3] H. H. Yeh, S. S. Banda and B. C. Chang (1992), Necessary and sufficient conditions for mixed H_2 and H_∞ optimal control, *IEEE Transactions on Automatic Control*, vol.37, no.3, pp. 355-358.

[4] P. P. Khargonekar and M. A. Rotea (1992), Mixed H_2/H_∞ control: A convex optimization approach, In *IEEE Transactions on Automatic Control*, vol.36, no.7, pp. 824-937.

[5] E. Baeyens and P. P. Khargonekar (1994), Some examples in mixed H_2/H_∞ control, In *Proc. ACC*, Baltimore, June, pp. 1608-1612.

[6] M. Green, D. Limebeer (1995), *Linear Robust Control*, Prentice-Hall.