Chapter 1

Introduction

Our primary interest in this thesis (and through the study of which we will later consider related problems in control and adaptive filtering) is estimation theory. Broadly speaking, in estimation theory one is confronted with the following problem: given the values of an observable signal (often called the measurement signal) one would like to estimate (or to predict) the values of another (so-called desired) signal that is not directly observable. Examples of where such problems may arise are numerous and are not difficult to conceive: in weather forecasting one has access to satellite measurements of current and past atmospheric pressure and humidity and would like to predict the future values of these quantities; in communications one typically observes (at the receiver) the output of a communication channel and would like to estimate the values of the bit stream sent (by the transmitter); and so on. In many applications, however, the estimation problem occurs in a more indirect fashion. One such application is in control theory where one observes the output of a dynamical system (the so-called plant) and where the goal is to influence the dynamics of the plant, through a control signal, so that the plant yields some desired behaviour. It turns out that it is often convenient to solve the control problem via a two-step procedure: one first uses the observed output of the plant to estimate the value of certain unobservable signals (that are internal to the plant), and then uses these estimates to construct the required control signals.

In view of the above, it is quite obvious that the solution to the problem of estimating an unobservable signal given an observable one depends on the relationship between the two signals (*i.e.*, on the model describing them) and on the criterion (*i.e.*, on the optimality principle) that one uses to determine the desired estimates. Of course, the above two issues are interrelated: one would like to choose a criterion that is compatible with the model, and vice-versa. However, what influence the choice of model and criterion most significantly are: the underlying problem that we are actually trying to solve (be it weather forecasting, control, or adaptive filtering), and the possibility of actually obtaining a solution to the formulated problem that is easily implementable. In other words, physical significance and mathematical tractability.

The subject of estimation theory, as covered by all possible choices of signal models and optimality criteria, is indeed a vast (and developing) one and is well beyond what can be studied in this thesis (or any other, for that matter). Therefore here (with some minor exceptions) our attention will be devoted to linear models (*i.e.*, cases where the measurement and desired signals are linearly related) and to quadratic (or quadratically-induced) deterministic and stochastic criteria. The linear models that we shall mostly be concerned with are those that are more relevant to system theory, *i.e.*, finite-dimensional linear state-space models and rational transfer matrices. The quadratic criteria that we shall consider include the (now classical) deterministic least-squares and stochastic least-mean-squares criteria, as well as the (more recent) ones in H^∞ theory, dynamic game theory and risk-sensitive estimation and control.

Indeed the major contribution of this thesis is that these apparently different estimation and control problems (with different deterministic and stochastic criteria) can be solved in a unified geometric framework using the concept of an indefinite inner-product (or indefinite metric) space. These so-called Krein spaces are extensions of Hilbert spaces where the self inner-product of any vector can be positive, negative or zero. The key observation is that many estimation problems can be reduced to a projection in a Krein space. Although Hilbert spaces and Krein spaces share many characteristics, they differ in special ways that turn out to mark the differences between the standard least-mean-squares (LQG or H^2) theories and the more recent H^∞ and game theories.

Apart from rather more transparent derivations of existing results, the major bonus of this unified approach is that it allows for many new results to be obtained by trying to extend to the H^∞, game-theoretic and risk-sensitive settings some of the huge body of results and insights developed over the last three decades in the fields of Kalman filtering and LQG control. In this thesis, this claim is backed by showing how to generalize the (possibly) numerically superior square-root algorithms and the (so-called) fast Chandrasekhar algorithms to these new settings, by performing new investigations on the asymptotic behaviour of H^∞ filters and controllers, and on the existence and properties of solutions to Riccati equations with (possibly) indefinite coefficient matrices. Moreover, this framework will be used to study the implications of robust estimation to the vast and highly active field of adaptive signal processing.

In this first introductory chapter we shall outline the major estimation and control problems to be studied and shall overview the scope and contributions of this thesis. The material of this chapter will also serve as a motivation for the study of indefinite metric spaces that will begin in Chapter 2. However, before doing so, it will be useful to present some very brief historical remarks.

1.1 Some Historical Remarks

The problem of interpreting observations and making estimates and predictions dates back to antiquity. Neugebauer [Neu57] has noted that the ancient Babylonians used a rudimentary form of the Fourier transform for such purposes. The beginnings of a theory of estimation, in the sense that one attempts to minimize a certain function of the errors, is apparently attributed to Galileo Galilei in 1632 [Gal32], after which one encounters a series of illustrious investigators including Roger Cotes, Euler, Lagrange, Laplace, Bernoulli and others.

The method of least-squares (for solving over-determined systems of linear equations), that chooses estimates that best match the observations in a least-squares sense, was apparently first used by Gauss in 1795 [Gau09], although first published by Legendre in 1805 [Leg10] and, independently, by Adrain in 1808 [Adr08]. Since then a vast literature has been developed both on deterministic least-squares problems (see, *e.g.*, any standard textbook on linear algebra and matrix analysis such as

[GL89], [HJ90] and [Str93]) and on least-squares estimation for random variables (see [Har72] for a comprehensive annotated bibliography on this subject).

The problem of least-mean-squares estimation of stochastic processes was first investigated by Kolmogorov [Kol39], [Kol41], Krein [Kre45a], [Kre45b] and Wiener [Wie49]. Although Kolmogorov's approach was more fundamental, the work of Wiener, especially the Wiener filter for the prediction of stationary stochastic processes, has turned out to be more influential. The most important contribution of the work of Kolmogorov and Wiener has been the introduction of statistical ideas to problems in estimation and control. In this framework the underlying signals, (*i.e.*, the measurement and desired signals) are assumed to be stochastic processes with known statistical properties (in particular, they are taken to be stationary stochastic processes with known first and second order statistics). The criterion for finding the desired estimate is then the least-mean-squares criterion, *i.e.*, the resulting signal estimates yield the smallest average squared estimation errors.

As noted above, the assumption that the underlying measurement and desired signal processes are stationary is crucial to the Wiener and Kolmogorov theory and it was not until the late 1950's and early 1960's that a satisfactory theory was developed, primarily by Kalman, that could treat the nonstationary case [Kal60b], [KB61] and [Kal63b]. The theory arose because of the inadequacy of the Wiener-Kolmogorov theory for coping with certain applications in which nonstationarity of the measurement and desired signals was intrinsic to the problem. The new theory soon acquired the name *Kalman filter* (or Wiener-Kalman filter) *theory*, and since then a vast literature on the topic has been developed. (An excellent survey of the developments up until the mid 1970's is given in [Kai74].)

Concurrent with the development of Kalman filter theory a closely related theory of optimal control was being developed in the United States and the Soviet Union [Kal60a], [KB60], [Kal64], [Pon61], [Yak62], [Pop64] and [Won68b]. As in the Kalman filter theory, the underlying assumptions of this theory were that the plant has a known linear (and possibly time-varying) description, and that the exogenous signals (the noises and disturbances) impinging on the feedback system are stochastic in nature, but have known statistical properties. These assumptions turned out to be very well suited to the problems of guidance and control of space vehicles to which the theory was first applied. This theory is now known as linear-quadratic-Gaussian (LQG) control to reflect the fact that the model and optimal controller are linear, that the cost function is quadratic, and that the disturbances are assumed to be stochastic processes with jointly Gaussian distribution.

As described above, classical methods in estimation theory (such as least-mean-squares, Wiener-Kalman, maximum-likelihood and maximum entropy) assume perfect models and regard the underlying signals as stochastic processes with known statistical properties. In many applications, however, one is faced with modeling errors and lack of statistical information. Therefore the aforementioned methods are not directly applicable since the statistics and distributions of the stochastic processes are not known. Moreover, it is not obvious what the behavior of such estimation schemes will be once the assumptions on the statistics and distributions are not met. This has led researches to consider *robust estimation theory* where the objective is to design estimators that have acceptable performance in the face of such deficiencies.

One approach that has been developed to address the above problem is (so-called) H^∞ estimation theory which has followed some pioneering work by Zames [Zam81] in robust control theory. (Some recent papers on H^∞ estimation include [KN91], [Bas91], [ST92] and [Gri93]). Robust control theory itself grew out of the need for designing controllers that were insensitive to plant modeling errors and to lack of statistical information on the exogenous signals. (In the late 1970's it was observed that LQG controllers could be highly nonrobust with respect to such modeling errors.) The H^∞ approach to robust control was extensively studied in the 1980's and has since been solved by numerous authors using various interpolation-theoretic and game-theoretic techniques [ZF83], [FZ84], [BC87], [Kim87], [DGKF89], [Tad90] and [GGLD90].

The main idea in H^∞ estimation is to come up with estimators that minimize (or in the suboptimal case, bound) the maximum energy gain from the disturbances to the estimation errors. This will guarantee that if the disturbances are small (in energy) then (no matter what the disturbances are) the estimation errors will be as small as possible (in energy). The robustness of H^∞ estimators, with respect to disturbance variation, follows from the fact that they safeguard against the estimators *worst-case* performance and make no assumptions on the statistics or distributions of the disturbance signals. Of course, since they make no such assumptions about the disturbances, they have to accommodate for all conceivable disturbances, and thus may be over-conservative.

Despite their fundamentally different objectives, the controllers and estimators obtained in H^∞ theory bear a striking resemblance to those obtained in LQG and Kalman filter theory. Nevertheless, there are enough significant differences that various ingenious methods have been devised to solve these H^∞ problems. Starting with the next chapter in this thesis, we will show that such very different solution methods need not be necessary; the basic LQG and Kalman filtering arguments can still be used, provided we set up appropriate control and estimation problems with elements not in a Hilbert space, but in an indefinite metric (so-called) Krein space. This observation has several different ramifications which we will also explore.

Finally, we should also mention some related developments that are somewhat to the periphery of what was explained above. Motivated primarily by econometric considerations, a game-theoretic approach to control and estimation was developed in the 1960's and 1970's [Isa65], [BO82] in which the disturbance signals are treated as an adversary player in a non-cooperative game. Also, theories for linear-exponential-quadratic-Gaussian (LEQG) and risk-sensitive optimal control and estimation have been developed in the 1970's and 1980's [Jac73], [SDJ74], [Whi81] that essentially replace the quadratic cost of LQG control with an exponential one. Both these theories turn out to be intimately related to H^∞ control and estimation, as has been noted in [GD88], [GM89], [Bas89] and [LAKG92], and as will be seen later in this thesis.

Estimation theory has, of course, much overlap with the fields of adaptive filtering, adaptive signal processing and adaptive neural networks [WS85], [Hay96], [RM86] and [Hay94]. However, even a brief survey of the developments in these related fields will take us too far from our current objectives. Therefore we shall defer an introduction to these areas until we treat them in Chapters 9 and 11.

1.2 A Basic Estimation Problem

A general discrete-time estimation problem is shown in Fig. 1.1.[1] Almost all estimation problems (such as Wiener, Kalman and adaptive filtering) can be cast into this framework. Here we assume that $\mathcal{H}$ and $\mathcal{L}$ are known *causal* linear transfer operators (or causal linear systems) that map the input sequence $\{u_j\}$ to their respective outputs. Although we shall not be specific about $\mathcal{H}$ and $\mathcal{L}$ here, we mention that in the finite horizon case $\mathcal{H}$ and $\mathcal{L}$ can be represented by finite block *lower triangular* matrices, and that in the infinite horizon case they are infinite (or semi-infinite) block lower triangular matrices. Another important instance is the infinite horizon case when $\mathcal{H}$ and $\mathcal{L}$ are time-invariant transfer operators, in which case we can represent them by transfer matrices (or transfer functions, in the scalar case) in the z-domain, namely $H(z)$ and $L(z)$. The model considered below is general and applies to all of the above cases.

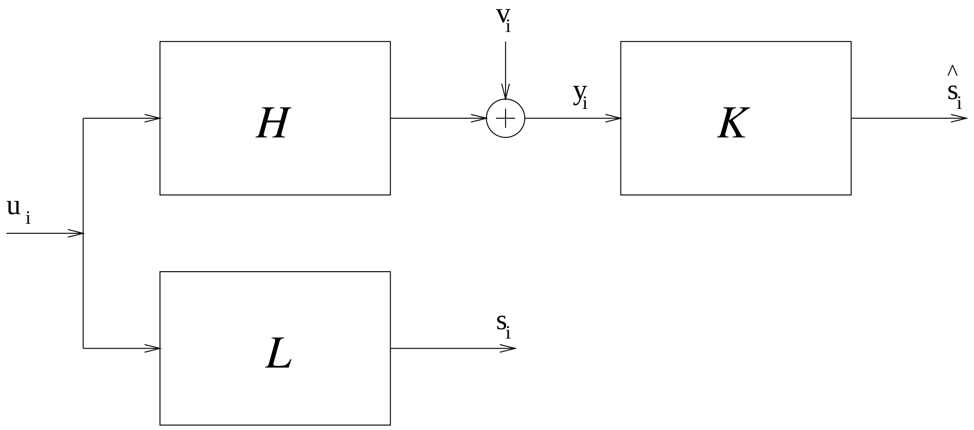

Figure 1.1: A general estimation problem.

In what follows we shall denote sequences such as $\{u_j\}$ by u, and simply write

$$s = \mathcal{L}u, \tag{1.2.1}$$

[1] In this thesis we shall, for the most part, be concerned with discrete-time estimation and control problems. Continuous-time counterparts of all the results presented are possible, and in most cases quite straightforward.

to denote that $\mathcal{L}$ maps the input sequence $\{u_j\}$ to the output sequence $\{s_j\}$.

The sequences $\{u_j\}$ and $\{v_j\}$ are assumed to be *unknown*.[2] [$\{u_j\}$ may be considered as a driving disturbance and $\{v_j\}$ as a measurement disturbance. In general, both may include modeling errors resulting from our lack of knowledge of the "true" $\mathcal{H}$ and $\mathcal{L}$.] The goal is to design a *causal* transfer operator (or filter) $\mathcal{K}$ that estimates s_i, the *unobservable* output of $\mathcal{L}$, using the *observations* $\{y_j, j \leq i\}$ (which can be regarded as corrupted measurements of the output of $\mathcal{H}$). The estimates we shall denote by $\hat{s}_{i|i}$ and the estimation errors by $\tilde{s}_{i|i} \triangleq s_i - \hat{s}_{i|i}$.

At this point let us note that (roughly speaking) the behavior of any estimator $\mathcal{K}$ can be captured by, $\mathcal{T}_{\mathcal{K}}$, the induced transfer operator that maps the unknown disturbances $\{u_j\}$ and $\{v_j\}$ to the estimation errors $\{\tilde{s}_{j|j}\}$. Thus

$$\mathcal{T}_{\mathcal{K}} : \begin{bmatrix} u \\ v \end{bmatrix} \rightarrow \tilde{s}. \tag{1.2.2}$$

Now using Fig. 1.1, we may write

$$\tilde{s} = s - \hat{s} = (\mathcal{L} - \mathcal{K}\mathcal{H})u - \mathcal{K}v = \begin{bmatrix} \mathcal{L} - \mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

from which we infer that

$$\mathcal{T}_{\mathcal{K}} = \begin{bmatrix} \mathcal{L} - \mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix}. \tag{1.2.3}$$

1.2.1 Special Cases

We now consider some special cases of the above general formulation.

Transfer Matrices

In the infinite-horizon case, when $\mathcal{H}$ and $\mathcal{L}$ are linear time-invariant transfer operators, they can be represented by transfer matrices, $H(z)$ and $L(z)$, of dimensions $p \times m$ and $q \times m$, respectively (assuming that the u_i, y_i and s_i are m-vectors, p-vectors and

[2] For the time being, we are purposefully ambiguous as to whether the $\{u_j\}$ and $\{v_j\}$ are deterministic or stochastic.

q-vectors, respectively). In this case we can write

$$\begin{cases} y(z) &= H(z)u(z) + v(z) \\ s(z) &= L(z)u(z) \end{cases} . \tag{1.2.4}$$

Assuming that the estimator $\mathcal{K}$ has a transfer matrix representation, $K(z)$, (consequently of dimension $q \times p$), then $\mathcal{T}_{\mathcal{K}}$ itself has the following transfer matrix representation

$$T_K(z) = \begin{bmatrix} L(z) - K(z)H(z) & -K(z) \end{bmatrix}. \tag{1.2.5}$$

State-Space Models

For a variety of reasons, it is often convenient to represent the relationship between the measurement signal, y_i, the desired signal, s_i, and the process and measurement noise signals, u_i and v_i, via a (possibly time-varying) linear state-space model. In this case, we can write

$$\begin{cases} x_{i+1} &= F_i x_i + G_i u_i \\ y_i &= H_i x_i + v_i \\ s_i &= L_i x_i \end{cases} , \tag{1.2.6}$$

where $F_i \in \mathcal{C}^{n\times n}$, $G_i \in \mathcal{C}^{n\times m}$, $H_i \in \mathcal{C}^{p\times n}$ and $L_i \in \mathcal{C}^{q\times n}$ are known system matrices, and where x_i is the n-dimensional state. Note that we have not specified the range of the time index, i, in (1.2.6) since the estimation problem may be finite, semi-infinite, or infinite horizon. [Note, moreover, that according to (1.2.6), since y_i and s_i depend on $\{u_j, j < i\}$, the transfer operators $\mathcal{H}$ and $\mathcal{L}$ are *strictly* causal. It turns out that there is no loss of generality in making this assumption. The benefit is that the algebraic expressions obtained are simpler.]

If we assume that the system matrices in (1.2.6) are time-invariant, *i.e.*,

$$F_i \triangleq F \quad , \quad G_i \triangleq G \quad , \quad H_i \triangleq H \quad , \quad L_i \triangleq L$$

then in the infinite-horizon case we can readily find the transfer matrices $H(z)$ and $L(z)$ from the system matrices via

$$\begin{cases} H(z) &= H(zI - F)^{-1}G \\ L(z) &= L(zI - F)^{-1}G \end{cases} . \tag{1.2.7}$$

Adaptive Filtering

In adaptive filtering we observe an output sequence $\{d_i\}$ that obeys the *linear* model

$$d_i = h_i^T w + v_i, \qquad i \geq 0 \tag{1.2.8}$$

where $h_i^T = \begin{bmatrix} h_{i1} & h_{i2} & \dots & h_{in} \end{bmatrix}$ is a known input vector, $w = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix}$ is an unknown weight vector, and $\{v_i\}$ is an unknown disturbance, which may also include modeling errors. The goal is to estimate some linear combination of the unknown weight vector, $L_i w$, (typically either with, $L_i = I$, for estimating w itself, or with, $L_i = h_i^T$, for estimating the uncorrupted output of the filter) using the observations, $\{d_j\}_{j=0}^i$.

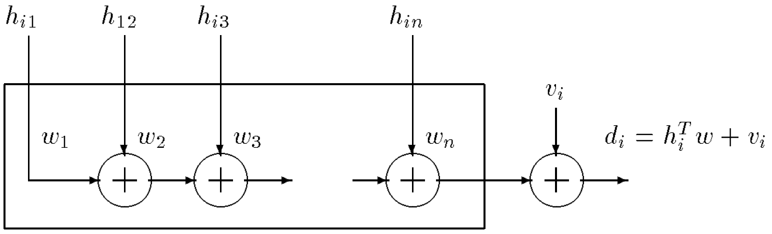

Figure 1.2: The model for adaptive filtering.

Comparing with the general estimation problem considered at the beginning of this section, we see that we can readily identify the observations $\{y_i\}$ with $\{d_i\}$, and that now the sequence $\{u_i\}$ is simply a constant n-dimensional vector, w. In this case it is straightforward to see that

$$\mathcal{H} = \begin{bmatrix} h_0^T \\ h_1^T \\ h_2^T \\ \vdots \end{bmatrix} \quad \text{and} \quad \mathcal{L} = \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ \vdots \end{bmatrix}. \tag{1.2.9}$$

A much more useful representation of the adaptive filtering problem is as a special case of a state-space estimation problem. (This point of view has been proposed and pursued in [SK94b] with great effect.) Indeed it is straightforward to see that we may write

$$\begin{cases} x_{i+1} &= x_i \\ d_i &= h_i^T x_i + v_i \\ s_i &= L_i x_i \end{cases}, \quad i \geq 0, \quad x_0 = w. \tag{1.2.10}$$

1.3 The H^2 Approach

The problem of estimation is to select $\mathcal{K}$, and thereby the estimates $\hat{s}_{i|i}$, based on some performance criterion. The most widely used of such criteria is the H^2 norm of the transfer operator, i.e., $\|\mathcal{T}_{\mathcal{K}}\|_2$.

(i) In the finite horizon case, $\|\mathcal{T}_{\mathcal{K}}\|_2$ is simply the Frobenius norm of the finite matrix $\mathcal{T}_{\mathcal{K}}$,

$$\|\mathcal{T}_{\mathcal{K}}\|_2 \triangleq \|\mathcal{T}_{\mathcal{K}}\|_F = (\text{trace}\,[\mathcal{T}_{\mathcal{K}}\mathcal{T}_{\mathcal{K}}^*])^{1/2} = \left(\sum_{i,j} \|T_{\mathcal{K},ij}\|_F^2\right)^{1/2}, \tag{1.3.1}$$

where $T_{\mathcal{K},ij}$ is the block (i,j)-th component of $\mathcal{T}_{\mathcal{K}}$, *i.e.*, $T_{\mathcal{K},ij}$ maps $\begin{bmatrix} u_j \\ v_j \end{bmatrix}$ to $\tilde{s}_{i|i}$.

(ii) In the infinite-horizon time-invariant case

$$\|\mathcal{T}_{\mathcal{K}}\|_2 \triangleq \left(\frac{1}{2\pi}\int_0^{2\pi} \|T_K(e^{j\omega})\|_F^2 d\omega\right)^{1/2}, \tag{1.3.2}$$

where $T_K(z)$ is now a transfer matrix.

The widespread use of the H^2 theory is mainly due to the facts that the optimal H^2 problem has a simple closed-form solution, and that, under certain statistical assumptions on the signals, the solution has several other desirable optimality properties.

Stochastic Interpretation

H^2-optimal estimators have the following two stochastic interpretations.

(a) Assume that the $\{u_j\}$ and $\{v_j\}$ are zero-mean, uncorrelated and temporally white stochastic processes with unit variance, *i.e.*,

$$E\begin{bmatrix} u_i \\ v_i \end{bmatrix}\begin{bmatrix} u_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}\delta_{ij}. \tag{1.3.3}$$

Consider the finite-horizon case (with time index i from 0 to N), and compute the estimation error energy,

$$\sum_{i=0}^{N}\tilde{s}_{i|i}^*\tilde{s}_{i|i} = \tilde{s}^*\tilde{s} = \begin{bmatrix} u^* & v^* \end{bmatrix}\mathcal{T}_{\mathcal{K}}^*\mathcal{T}_{\mathcal{K}}\begin{bmatrix} u \\ v \end{bmatrix} = \text{trace}\left(\mathcal{T}_{\mathcal{K}}^*\mathcal{T}_{\mathcal{K}}\begin{bmatrix} u \\ v \end{bmatrix}\begin{bmatrix} u^* & v^* \end{bmatrix}\right). \tag{1.3.4}$$

In view of (1.3.3), we have $E\begin{bmatrix} u \\ v \end{bmatrix}\begin{bmatrix} u^* & v^* \end{bmatrix} = I$, so that taking expectations from both sides of the above equation, the expected estimation error energy becomes

$$E\sum_{i=0}^{N}\tilde{s}_{i|i}^*\tilde{s}_{i|i} = \text{trace}\left(\mathcal{T}_{\mathcal{K}}^*\mathcal{T}_{\mathcal{K}}\right) = \|\mathcal{T}_{\mathcal{K}}\|_2^2. \tag{1.3.5}$$

But this is simply the cost function that H^2-optimal estimators minimize. Therefore, in the finite-horizon case, and under the aforementioned statistical assumptions, H^2-optimal estimators minimize the expected estimation error energy. This is why they are also referred to as *linear least-mean-squares estimators*.

Using a similar argument in the infinite-horizon time-invariant case, it is possible to show that

$$E\tilde{s}_{i|i}^*\tilde{s}_{i|i} = \frac{1}{2\pi}\int_0^{2\pi}\|T_K(e^{j\omega})\|_F^2 d\omega = \|\mathcal{T}_{\mathcal{K}}\|_2^2. \tag{1.3.6}$$

Therefore, in the infinite-horizon case, H^2-optimal estimators minimize the expected squared estimation error.

(b) If, in addition to the assumptions of part (a), the $\{u_j\}$ and $\{v_j\}$ are assumed to be jointly Gaussian, then the H^2-optimal estimator is a *least-mean-squares* estimator (*i.e.*, we do not need to restrict the estimator to being linear) and in addition yields the maximum-likelihood estimate of the $\{s_i\}$.

1.3.1 The General Solution

The solution to the H^2 estimation problem is well known (see *e.g.*, [Jaz70], [AM79] and [Kai81]) and is given in the following Theorem.

Theorem 1.3.1 (H^2-optimal Estimator) *The solution to the problem*

$$\min_{\text{causal } \mathcal{K}} \|\mathcal{T}_{\mathcal{K}}\|_2, \tag{1.3.7}$$

is given by

$$\mathcal{K} = \left\{\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}\right\}_+ (I+\mathcal{H}\mathcal{H}^*)^{-1/2}, \tag{1.3.8}$$

where $(I+\mathcal{H}\mathcal{H}^*)^{-1/2}$ and $(I+\mathcal{H}\mathcal{H}^*)^{-*/2}$ are found from the canonical (minimum-phase maximum-phase) factorization

$$I+\mathcal{H}\mathcal{H}^* = (I+\mathcal{H}\mathcal{H}^*)^{1/2}(I+\mathcal{H}\mathcal{H}^*)^{*/2}, \tag{1.3.9}$$

and where the notation $\{\mathcal{A}\}_+$ denotes the causal part of the transfer operator, $\mathcal{A}$.

In (1.3.9) the transfer operator $(I+\mathcal{H}\mathcal{H}^*)^{1/2}$ is both causal and causally invertible (hence minimum phase) and the transfer operator $(I+\mathcal{H}\mathcal{H}^*)^{-*/2}$ is both anti-causal and anti-causally invertible (hence maximum phase). Such a factorization of the positive-definite operator $I+\mathcal{H}\mathcal{H}^*$ always exists and is referred to as the *canonical factorization*.

In the finite-horizon case (1.3.9) is the LL^* (block lower-upper triangular) decomposition of the matrix $I+\mathcal{H}\mathcal{H}^*$, and the notation $\{\mathcal{A}\}_+$ denotes the (block) lower triangular part of the matrix $\mathcal{A}$. [Note in this case that the matrix $I+\mathcal{H}\mathcal{H}^*$ is the covariance matrix of the observations signal, y, and that therefore (1.3.9) is the canonical factorization of this covariance matrix.] In the time-invariant infinite-horizon case (1.3.9) is the spectral factorization of the z-spectral density $I+H(z)H^*(z^{-*})$, and $\{A(z)\}_+$ is the causal part of the function $A(z)$. [Note that in this case $I+\mathcal{H}\mathcal{H}^*$ is the z-spectral density function of the observations process, y.]

The proof of Theorem 1.3.1 is instructive and is presented below.

Proof of Theorem 1.3.1: First note that

$$\|\mathcal{T}_{\mathcal{K}}\|_2^2 = \mathrm{trace}(\mathcal{T}_{\mathcal{K}}^*\mathcal{T}_{\mathcal{K}}) = \mathrm{trace}(\mathcal{T}_{\mathcal{K}}\mathcal{T}_{\mathcal{K}}^*).$$

Using (1.2.3), the expression for $\mathcal{T}_{\mathcal{K}}$, we can write

$$\begin{aligned}\mathcal{T}_{\mathcal{K}}\mathcal{T}_{\mathcal{K}}^* &= (\mathcal{L}-\mathcal{K}\mathcal{H})(\mathcal{L}-\mathcal{K}\mathcal{H})^*+\mathcal{K}\mathcal{K}^* \\ &= \left[\mathcal{K}-\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-1}\right](I+\mathcal{H}\mathcal{H}^*)\left[\mathcal{K}-\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-1}\right]^*+\mathcal{L}(I+\mathcal{H}^*\mathcal{H})^{-1}\mathcal{L}^*\end{aligned}$$

where to obtain the second equality we have used a completion of squares argument. Now using the canonical factorization (1.3.9), and the linearity of the trace$(\cdot)$ operator, allows us to conclude

$$\|\mathcal{T}_{\mathcal{K}}\|_2^2 = \|\mathcal{K}(I+\mathcal{H}\mathcal{H}^*)^{1/2}-\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}\|_2^2+\|\mathcal{L}(I+\mathcal{H}^*\mathcal{H})^{-*/2}\|_2^2.$$

Note that only the first term on the RHS of the above equation depends on $\mathcal{K}$. Therefore it suffices to minimize this first term over causal transfer operators, $\mathcal{K}$. Further inspection of this first term reveals that although $\mathcal{K}(I+\mathcal{H}\mathcal{H}^*)^{1/2}$ is a causal operator, $\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}$ is not. However, using the readily verified identity

$$\|\mathcal{A}\|_2^2 = \|\{\mathcal{A}\}_+\|_2^2+\|\{\mathcal{A}\}_-\|_2^2\,,$$

where we have denoted the strictly anti-causal part of $\mathcal{A}$ by $\{\mathcal{A}\}_- \triangleq \mathcal{A}-\{\mathcal{A}\}_+$, we may write

$$\begin{gathered}\|\mathcal{K}(I+\mathcal{H}\mathcal{H}^*)^{1/2}-\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}\|_2^2 = \\ \left\|\mathcal{K}(I+\mathcal{H}\mathcal{H}^*)^{1/2}-\{\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}\}_+\right\|_2^2+\left\|\{\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}\}_-\right\|_2^2.\end{gathered}$$

Note, once more, that the second term on RHS is independent of $\mathcal{K}$. Choosing $\mathcal{K}$ according to (1.3.8) makes the first term vanish and obviously minimizes $\|\mathcal{T}_{\mathcal{K}}\|_2$.

■

Remark: The main conclusion to be made from the above proof is that the solution to the H^2-optimal estimation problem is obtained from the canonical factorization of a positive definite transfer operator.

1.3.2 Special Cases

Depending on the nature of the transfer operators, $\mathcal{H}$ and $\mathcal{L}$, the H^2 optimal solution of Theorem 1.3.1 takes on various forms. When $\mathcal{H}$ and $\mathcal{L}$ have state-space structure the solution yields the Kalman filter. When they have transfer function representations, $H(z)$ and $L(z)$, the solution is the Wiener filter, and in the adaptive filtering case the solution corresponds to the recursive-least-squares (RLS) algorithm. We shall now briefly present these.

The Wiener Filter

As noted in Sec. 1.2.1, in the Wiener filtering problem the model for the measurement and desired signals is given by

$$\left\{ \begin{array}{rcl} y(z) & = & H(z)u(z) + v(z) \\ s(z) & = & L(z)u(z) \end{array} \right. , \tag{1.3.10}$$

where $\{u_i\}$ and $\{v_i\}$ are assumed to be zero-mean uncorrelated and temporally white stationary processes such that[3]

$$E \begin{bmatrix} u_i \\ v_i \end{bmatrix} \begin{bmatrix} u_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}, \tag{1.3.11}$$

or, equivalently,

$$S_u(z) = Q \quad , \quad S_v(z) = R \quad , \quad S_{uv}(z) = 0 \tag{1.3.12}$$

where $S_u(z)$, $S_v(z)$ and $S_{uv}(z)$ are the (obvious) z-spectral and cross z-spectral densities of the stationary stochastic processes, $\{v_i\}$ and $\{v_i\}$.

The Wiener filter for causally estimating s_i, using the $\{y_j, j \leq i\}$, follows straightforwardly from Theorem 1.3.1 and is given by

$$K(z) = \left\{ L(z)QH^*(z^{-*})M^{-*}(z^{-*}) \right\}_+ R_e^{-1} M^{-1}(z), \tag{1.3.13}$$

where $M(z)$, the so-called *modeling filter*, is found from the canonical spectral factorization

$$R + H(z)QH^*(z^{-*}) = M(z)R_e M^*(z^{-*}), \tag{1.3.14}$$

with $M(z)$ causal and causally invertible, and where R_e is a constant matrix chosen such that we have the normalization

$$M(\infty) = I_p. \tag{1.3.15}$$

[3]It is straightforward to consider the case where u_i and v_i are correlated, but for simplicity we shall assume the uncorrelated case here.

Therefore, to find the Wiener filter, all we need to do is perform the canonical spectral factorization (1.3.14). When the z-power spectral density function is a scalar, the factorization in (1.3.14) is straightforward, and is obtained by retaining the stable (inside the unit circle) poles and zeros of $R + H(z)QH^*(z^{-*})$ for $M(z)$ (see *e.g.*, [Kai81]). When $R+H(z)QH^*(z^{-*})$ is a matrix, computing the canonical factorization is much more involved and requires the Smith-McMillan form [You61], [Yak70a] and [Kai80].

However, when the transfer matrices have state-space structure, viz.,

$$\begin{cases} H(z) &= H(zI-F)^{-1}G \\ L(z) &= L(zI-F)^{-1}G \end{cases}, \tag{1.3.16}$$

(recall (1.2.7)), then the canonical factorization can be found via solving a *discrete-time algebraic Riccati equation* (DARE) [Wil71b]. Indeed, if $\{F, GQ^{1/2}\}$ is stabilizable[4] and $\{F, H\}$ is detectable[5], then the modeling filter in (1.3.14) is given by

$$M(z) = I_p + H(zI-F)^{-1}K_p \quad , \quad K_p = FPH^*R_e^{-1} \quad , \quad R_e = R + HPH^* \tag{1.3.17}$$

where P is the unique positive semidefinite solution to the DARE

$$P = FPF^* + GQG^* - K_pR_eK_p^*. \tag{1.3.18}$$

Moreover, P is such that the matrix,

$$F_p \triangleq F - K_pH, \tag{1.3.19}$$

is stable, which is in accordance with the fact that the inverse of the modeling filter,

$$M^{-1}(z) = I - H(zI - F + K_pH)^{-1}K_p = I - H(zI-F_p)^{-1}K_p, \tag{1.3.20}$$

must be causal.

[4]This is a system-theoretic concept. The pair $\{F, GQ^{1/2}\}$ is called stabilizable if there exists a matrix K such that $F - GQ^{1/2}K$ is stable, *i.e.*, if F can be stabilized through state feedback (through the input Gu) [Kai80].

[5]Detectability is the dual concept to stabilizability. The pair $\{F, H\}$ is detectable if $\{F^*, H^*\}$ is stabilizable, *i.e.*, if there exists a matrix K such that $F - KH$ is stable.

We are now in a position to give a more explicit formula for the Wiener filter, $K(z)$, of (1.3.13). To this end, note that

$$\begin{aligned} H^*(z^{-*})M^{-*}(z^{-*}) &= G^*(z^{-1}I - F^*)^{-1}H^*\left[I_p - K_p^*(z^{-1}I - F^*)^{-1}H^*\right]^{-1} \\ &= G^*(z^{-1}I - F^* + H^*K_p^*)^{-1}H^* \\ &= G^*(z^{-1}I - F_p^*)^{-1}H^*, \end{aligned}$$

so that we may write

$$\left\{L(z)QH^*(z^{-*})M^{-*}(z^{-*})\right\}_+ = \left\{L(zI-F)^{-1}GQG^*(z^{-1}-F_p^*)^{-1}H^*\right\}_+.$$

The above expression shows that to find $K(z)$ we need to find the causal part of the transfer matrix, $(zI-F)^{-1}GQG^*(z^{-1}-F_p^*)^{-1}$. But here is a little trick to do so.[6] Using the DARE we may replace GQG^* by

$$P - FPF^* + K_pR_eK_p^* = P - FPF^* + FPH^*K_p^* = P - FP(F-K_pH)^* = P - FPF_p^*,$$

and write

$$(zI-F)^{-1}GQG^*(z^{-1}-F_p^*)^{-1} = (zI-F)^{-1}(P-FPF_p^*)(z^{-1}-F_p^*)^{-1}.$$

Now replacing the center matrix, $P - FPF_p^*$, by

$$P - (zI - (zI-F))\,P\left(z^{-1}I - (z^{-1}-F_p^*)\right),$$

we get, after some algebraic simplifications, that

$$(zI-F)^{-1}GQG^*(z^{-1}-F_p^*)^{-1} = P + (zI-F)^{-1}FP + PF_p^*(z^{-1}-F_p^*)^{-1}. \quad (1.3.21)$$

The above expression is the desired decomposition of $(zI-F)^{-1}GQG^*(z^{-1}-F_p^*)^{-1}$ into its causal and anticausal parts. Indeed since F is stable (by assumption) and F_p is stable (by the solution to the DARE), we have

$$\underbrace{P + \underbrace{(zI-F)^{-1}FP}_{\text{strictly causal}}}_{\text{causal}} + \underbrace{PF_p^*(z^{-1}-F_p^*)^{-1}}_{\text{strictly anticausal}}.$$

[6]We shall see the origin of this trick later in Chapter 7.

This then allows us to write

$$\left\{L(z)QH^*(z^{-*})M^{-*}(z^{-*})\right\}_+ = LPH^* + L(zI-F)^{-1}FPH^*, \tag{1.3.22}$$

so that using (1.3.13) we finally obtain the desired expression for $K(z)$,

$$\begin{aligned} K(z) &= \left(LPH^* + L(zI-F)^{-1}FPH^*\right) R_e^{-1}M^{-1}(z) \\ &= LPH^*R_e^{-1} + L(I-PH^*R_e^{-1}H)(zI-F_p)^{-1}K_p. \end{aligned} \tag{1.3.23}$$

where to obtain the second equality we have used the expression for $M^{-1}(z)$ from (1.3.20).

We can now use the second expression in (1.3.23) to write down a state-space model for the Wiener filter as follows:

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1} &=& (F-K_pH)\hat{x}_i + K_p y_i \\ \hat{s}_{i|i} &=& L(I-PH^*R_e^{-1}H)\hat{x}_i + LPH^*R_e^{-1}y_i \end{array} \right. , \tag{1.3.24}$$

where the "hat" notation in the state variable $\hat{x}(z) \triangleq (zI-F_p)^{-1}K_p y(z)$, has been used since it turns out that $\hat{x}_i$ is indeed the least-means squares prediction of the original state $\hat{x}_i$, given the observations, $\{y_j, j<i\}$.

A further definition that is useful is the so-called *innovations process* [Kai68],

$$e(z) \triangleq M^{-1}(z)y(z) = \left[I_p - H(zI-F_p)^{-1}K_p\right] y(z) = y(z) - H\hat{x}(z), \tag{1.3.25}$$

which, as can be readily verified from the spectral factorization (1.3.14), is a white stationary stochastic process with variance, R_e.

It is useful to summarize the results obtained so far in the following Theorem.

Theorem 1.3.2 (Wiener Filter) *The solution to the problem*

$$\min_{\text{causal } K(z)} \left\| \left[\begin{array}{cc} (L(z)-K(z)H(z))\,Q^{1/2} & -K(z)R^{1/2} \end{array} \right] \right\|_2, \tag{1.3.26}$$

is given by

$$K(z) = \left\{L(z)QH^*(z^{-*})M^{-*}(z^{-*})\right\}_+ R_e^{-1}M^{-1}(z), \tag{1.3.27}$$

where $M(z)$ is found from the canonical spectral factorization

$$R + H(z)QH^*(z^{-*}) = M(z)R_eM^*(z^{-*}),$$

with $M(z)$ causal and causally invertible, and $M(\infty) = I_p$.

When $H(z)$ and $L(z)$ have state-space structure,

$$\begin{cases} H(z) &= H(zI-F)^{-1}G \\ L(z) &= L(zI-F)^{-1}G \end{cases},$$

with $\{F, GQ^{1/2}\}$ stabilizable and $\{F, H\}$ detectable, then

$$K(z) = LPH^*R_e^{-1} + L(I - PH^*R_e^{-1}H)(zI - F_p)^{-1}K_p, \tag{1.3.28}$$

where $K_p = FPH^*R_e^{-1}$, $R_e = R + HPH^*$, and P is the unique positive semidefinite solution of the DARE,

$$P = FPF^* + GQG^* - K_pR_eK_p^*.$$

In this case, a state-space model for $K(z)$ can be given by

$$\begin{cases} \hat{x}_{i+1} &= (F - K_pH)\hat{x}_i + K_py_i \\ \hat{s}_{i|i} &= L(I - PH^*R_e^{-1}H)\hat{x}_i + LPH^*R_e^{-1}y_i \end{cases}, \tag{1.3.29}$$

or, defining the innovations, $e_i = y_i - H\hat{x}_i$,

$$\begin{cases} \hat{x}_{i+1} &= F\hat{x}_i + K_pe_i \\ \hat{s}_{i|i} &= L\hat{x}_i + LPH^*R_e^{-1}e_i \end{cases}. \tag{1.3.30}$$

Remark: The filters (1.3.29) and (1.3.29) are the so-called *predicted* form of the Wiener filter (since the state is the predicted estimate, $\hat{x}_i$). Defining $\hat{x}_{i|i} \triangleq \hat{x}_i + PH^*R_e^{-1}e_i$, we can write the following so-called *filtered* form of the Wiener filter

$$\begin{cases} \hat{x}_{i+1|i+1} &= F\hat{x}_{i|i} + PH^*R_e^{-1}(y_{i+1} - HF\hat{x}_{i|i}) \\ \hat{s}_{i|i} &= L\hat{x}_{i|i} \end{cases}. \tag{1.3.31}$$

The Kalman Filter

As noted in Sec. 1.2.1, in the Kalman filtering problem the model for the measurement and desired signals is given by a (possibly time-variant) linear state-space model

$$\begin{cases} x_{i+1} &= F_ix_i + G_iu_i, \quad i \geq 0, \ x_0 \\ y_i &= H_ix_i + v_i \\ s_i &= L_ix_i \end{cases}. \tag{1.3.32}$$

Moreover, it is assumed that x_0 and the $\{u_i\}$ and $\{v_i\}$ are zero-mean uncorrelated random variables with known covariance matrices[7]

$$E\begin{bmatrix} x_0 \\ u_i \\ v_i \end{bmatrix}\begin{bmatrix} x_0^* & u_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & 0 \\ 0 & 0 & R_i\delta_{ij} \end{bmatrix}. \tag{1.3.33}$$

In this case, $\hat{s}_{i|i}$, the causal linear least-mean-squares estimate of s_i, is given by the Kalman filter recursions [Kal60b],

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1} & = & F_i\hat{x}_i + K_{p,i}e_i, \quad \hat{x}_0 = 0 \\ \hat{s}_{i|i} & = & L_i\hat{x}_i + L_iP_iH_i^*R_{e,i}^{-1}e_i \end{array} \right. , \tag{1.3.34}$$

where $e_i \triangleq y_i - H_i\hat{x}_i$ is the (white) innovations process, and where we have defined

$$K_{p,i} = F_iP_iH_i^*R_{e,i}^{-1} \quad \text{and} \quad R_{e,i} = R_i + H_iP_iH_i^* \tag{1.3.35}$$

with P_i the solution to the Riccati recursion

$$P_{i+1} = F_iP_iF_i^* + G_iQ_iG_i^* - K_{p,i}R_{e,i}K_{p,i}^*, \quad P_0 = \Pi_0. \tag{1.3.36}$$

Remarks:

(i) Although not shown explicitly here, the Kalman filter recursively performs the (block) triangular decomposition of the output Gramian matrix, $I + \mathcal{H}\mathcal{H}^*$, via the Riccati recursion (1.3.36). (Recall that the Wiener filter performed the canonical factorization of the output z-spectral density via the solution of an algebraic Riccati equation). We shall show this result, in fact, in the much more general context of indefinite metric spaces (and indefinite output Gramians) in the next chapter.

(ii) When the $\{F_i, G_i, H_i, Q_i, R_i\}$ are constant matrices, the Kalman filter recursions (1.3.34) bear a striking resemblance to the Wiener filter of Theorem 1.3.2. Indeed it is true that in the time-invariant case, under some rather mild conditions, the Kalman filter recursions converge to the Wiener filter. We will have much more to say about this in Chapter 8.

[7]Once more, for simplicity, we shall assume that the u_i and v_i are uncorrelated.

(iii) The Kalman filter recursions (1.3.34) are in so-called predicted form. Defining the filtered estimates of the states as $\hat{x}_{i|i} \triangleq \hat{x}_i + P_i H_i^* R_{e,i}^{-1} e_i$, we obtain the filtered form of the Kalman filter recursions

$$\begin{cases} \hat{x}_{i+1|i+1} &= F_i \hat{x}_{i|i} + P_i H_i^* R_{e,i}^{-1}(y_{i+1} - H_{i+1} F_i \hat{x}_{i|i}) \quad \hat{x}_{-1|-1} = 0 \\ \hat{s}_{i|i} &= L_i \hat{x}_{i|i} \end{cases} . \tag{1.3.37}$$

(iv) The Kalman filter solution (as well as the Wiener filter solution) has the interesting property that the structure of the filter does not depend on the linear combination of the state that we intend to estimate (the Riccati recursion and the recursion for $\hat{x}_i$ do not depend on L_i). Therefore if one were interested in estimating some other linear combination of the state, say $s_i' = L_i' x_i$, the solution is simply that linear combination of the state estimate, *i.e.*, $\hat{s}_{i|i}' = L_i' \hat{x}_{i|i}$.

(v) There is now a vast literature on the Kalman filter and many variations to the recursions described so far have been developed. We mention in passing the square-root forms of these filters (see *e.g.*, [BG66], [DM69], [KBS71] and the references therein) and the fast (so-called) Chandrasekhar recursions for time-invariant (or structure time-variant) state-space models (see [Kai72], [MSK74] and [SK94a] and the references therein) which we shall encounter in a more general context in Chapter 5.

The RLS Algorithm

As noted in Sec. 1.2.1, the adaptive filtering problem can be recast as a special case of a state-space estimation problem where the state-space model has the form,

$$\begin{cases} x_{i+1} &= x_i \\ d_i &= h_i^T x_i + v_i \\ s_i &= L_i x_i \end{cases} , \quad i \geq 0, \quad x_0 = w. \tag{1.3.38}$$

i.e., $F_i = I$, $G_i = 0$ and $H_i = h_i^T$. The solution is consequently a special case of the Kalman filter and is given below

$$\begin{cases} \hat{s}_{i|i} &= L_i \hat{w}_{|i} \\ \hat{w}_{|i} &= \hat{w}_{|i-1} + \frac{P_i h_i}{1+h_i^T P_i h_i}(d_i - h_i^T \hat{w}_{|i-1}), \quad \hat{w}_{|-1} = 0 \end{cases} , \tag{1.3.39}$$

where $\hat{w}_{|i} = \hat{x}_{i|i}$, and where P_i satisfies the Riccati recursion

$$P_{i+1} = P_i - \frac{P_i h_i h_i^T P_i}{1 + h_i^T P_i h_i}, \quad P_0 = \Pi_0. \tag{1.3.40}$$

In the adaptive filtering literature, the algorithm (1.3.39-1.3.40) is known as the *recursive least-squares* (RLS) algorithm [Hay96].

1.3.3 The Question of Robustness

We saw that under suitable stochastic assumptions, H^2-optimal estimators have certain desirable optimality properties, namely that they minimize the expected estimation error energy and yield maximum-likelihood estimates.

Since, in practice we may not always know the statistics of the disturbances we cannot always guarantee the validity of the assumptions required of H^2 estimators. Therefore, the question that begs itself is what the performance of such estimators will be if the assumptions on the disturbances are violated, or if there are modeling errors in our model so that the disturbances must include the modeling errors? In other words

- *is it possible that* **small** *disturbances and modeling errors may lead to* **large** *estimation errors?*

Intuitively, a non-robust algorithm is one for which the above is true, *i.e.*, one for which small disturbances may lead to large estimation errors, and a robust algorithm is one for which small disturbances lead to small estimation errors.

The problem of robust estimation is thus an important one. As we shall presently see, the H^∞ estimation formulation is an *attempt* at addressing this question. It follows from our comments above that any approach to robust estimation requires a *measure* of largeness and smallness (for the signals involved), and in the H^∞ framework this measure is *energy*[8]. The idea is to come up with estimators that minimize (or in the suboptimal case, bound) the maximum energy gain from the disturbances to the estimation errors. This will guarantee that if the disturbances are small (in

[8]Other approaches to robust estimation and control differ in how this measure is defined. For example, in the so-called l_1 approach to robust control the measure used is the peak (maximum of the absolute amplitude) of the signals [DP87] and [DDB95].

energy) then the estimation errors will be as small as possible (in energy), *no matter what the disturbances are*. In other words the maximum energy gain is minimized over *all possible* disturbances. The robustness of the H^∞ estimators arises from this fact. However, since they make no assumption about the disturbances and have to accommodate for all conceivable ones, they may be over-conservative.

1.4 The H^∞ Approach

In this section we briefly describe the H^∞ approach to robust estimation. [For alternative presentations and derivations see [Kwa86], [DGKF89], [KN91], [Bas91], [LS91], [ST92], [Gri93] and the references therein.]

Returning to the general estimation problem of Sec. 1.2, we recall that a useful representation for any estimation strategy $\mathcal{K}$ is the transfer operator,

$$\mathcal{T}_{\mathcal{K}} = \begin{bmatrix} \mathcal{L} - \mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix},$$

that maps the disturbance sequences $\{u_j\}$ and $\{v_j\}$ to the estimation error sequence $\{\tilde{s}_{j|j}\}$. Now for any disturbance sequences u and v that yield the estimation error sequence $\tilde{s}$, we may compute the energy gain

$$\frac{\|\tilde{s}\|_2^2}{\|u\|_2^2 + \|v\|_2^2} = \frac{\left\| \mathcal{T}_{\mathcal{K}} \begin{bmatrix} u \\ v \end{bmatrix} \right\|_2^2}{\|u\|_2^2 + \|v\|_2^2}, \tag{1.4.1}$$

where $\|a\|_2^2 = \sum_j a_j^* a_j$ is defined as the energy of the sequence $a = \{a_j\}$.[9] Thus (1.4.1) is a measure of the "amplification" of the noise given our choice of estimator $\mathcal{K}$. Clearly, the ratio in (1.4.1) depends on the particular choice of the input disturbances u and v. To remove this dependence we consider the largest energy gain in (1.4.1) over *all* possible disturbance sequences u and v, i.e., the H^∞ norm of a transfer operator $\mathcal{T}_{\mathcal{K}}$, as defined below.

[9]Note that we are using the same notation, $\|\cdot\|_2$, for the two-norm of a sequence (the square-root of the energy) and the two norm of an operator. Which one will be meant will be obvious from the context throughout.

Definition 1.4.1 (The H^∞ Norm) *The H^∞ norm of a transfer operator $\mathcal{T}$ is defined as*

$$\|\mathcal{T}\|_\infty = \sup_{x \in h^2, x \neq 0} \frac{\|\mathcal{T}x\|_2}{\|x\|_2} \tag{1.4.2}$$

where h^2 denotes the space of all square-summable causal sequences.

(i) In the finite horizon case, $\|\mathcal{T}\|_\infty$ is simply $\bar{\sigma}(\mathcal{T})$, the maximum singular value of $\mathcal{T}$.

(ii) In the infinite-horizon time-invariant case, we have

$$\|\mathcal{T}\|_\infty = \sup_{0 \leq \omega \leq 2\pi} \bar{\sigma}(T(e^{j\omega})), \tag{1.4.3}$$

which is really the origin of the name H^∞.

In H^∞ estimation one seeks the causal estimator $\mathcal{K}$ that minimizes the H^∞ norm of $\mathcal{T}_\mathcal{K}$. The precise statement of the problem follows.

Problem 1.4.1 (Optimal H^∞ Estimation Problem) *Find a causal estimator $\mathcal{K}$ that minimizes the H^∞ norm of the transfer operator $\mathcal{T}_\mathcal{K}$ that maps the disturbances $\{u_j\}$ and $\{v_j\}$ to the estimation errors $\{\tilde{s}_{j|j}\}$,* i.e. , *find a causal $\mathcal{K}$ that satisfies,*

$$\inf_\mathcal{K} \|\mathcal{T}_\mathcal{K}\|_\infty = \inf_\mathcal{K} \left\| \begin{bmatrix} \mathcal{L} - \mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix} \right\|_\infty = \inf_\mathcal{K} \sup_{u,v \in h^2, u,v \neq 0} \frac{\left\| \mathcal{T}_\mathcal{K} \begin{bmatrix} u \\ v \end{bmatrix} \right\|_2}{(\|u\|_2^2 + \|v\|_2^2)^{1/2}}. \tag{1.4.4}$$

Moreover find the resulting $\gamma_{opt} = \inf_\mathcal{K} \|\mathcal{T}_\mathcal{K}\|_\infty$.

[Note that in the above problem statement we are deliberately ambiguous as to whether we are considering the finite horizon or (semi)infinite horizon case.]

The *minimax* nature of H^∞ optimal estimators is evident from (1.4.4). The H^∞ estimation problem can thus be regarded as a game problem: nature (the opponent) has access to the unknown disturbance sequences u and v and chooses it to maximize the energy gain in (1.4.1), whereas we have choice of the causal estimator $\mathcal{K}$ and must choose it to minimize the ratio in (1.4.1).

Note that H^∞ optimal estimators safeguard against the worst-case disturbance that maximizes the energy gain to estimation errors. Since this worst-case disturbance is a single event, such estimators do not require any statistical assumptions on the disturbance signals. Moreover, since the minimization in (1.4.4) is taken over all possible disturbances, these algorithms are robust with respect to disturbance variation.

Unlike in H^2 estimation, there are very few cases where a closed-form solution to the optimal H^∞ problem of Prob. 1.4.1 can be found,[10] and in general one relaxes the minimization and settles for a suboptimal solution.

Problem 1.4.2 (Suboptimal H^∞ Estimation Problem) *Given a $\gamma > 0$, find a causal estimator $\mathcal{K}$ that guarantees*

$$\|\mathcal{T}_{\mathcal{K}}\|_\infty = \left\| \begin{bmatrix} \mathcal{L} - \mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix} \right\|_\infty = \sup_{u,v \in h^2, u,v \neq 0} \frac{\left\| \mathcal{T}_{\mathcal{K}} \begin{bmatrix} u \\ v \end{bmatrix} \right\|_2}{(\|u\|_2^2 + \|v\|_2^2)^{1/2}} < \gamma. \tag{1.4.5}$$

This clearly requires checking whether $\gamma \geq \gamma_{opt}$.

1.4.1 The General Solution

We now outline how to find suboptimal H^∞ estimators that achieve a certain level γ. First note that $\|\mathcal{T}_{\mathcal{K}}\|_\infty < \gamma$ means that

$$\mathcal{T}_{\mathcal{K}} \mathcal{T}_{\mathcal{K}}^* < \gamma^2 I, \tag{1.4.6}$$

where I is the identity transfer operator that maps input sequences to themselves. In other words, the transfer operator, $\gamma^2 I - \mathcal{T}_{\mathcal{K}} \mathcal{T}_{\mathcal{K}}^*$, must be positive definite. [When $\mathcal{T}_{\mathcal{K}}$ is a matrix, the inequality in (1.4.6) is understood to be in the sense of the ordering of Hermitian positive semi-definite matrices, and when $\mathcal{T}_{\mathcal{K}}$ can be represented as a transfer matrix, $T(z)$, it is understood to be in the sense of $T(e^{j\omega})T^*(e^{j\omega}) < \gamma^2 I$, for all $0 \leq \omega \leq 2\pi$.]

[10] One such case is adaptive filtering which we shall study in detail in Chapter 9.

In either case, (1.4.6) can be rewritten as

$$(\mathcal{L}-\mathcal{K}\mathcal{H})(\mathcal{L}-\mathcal{K}\mathcal{H})^*+\mathcal{K}\mathcal{K}^*-\gamma^2 I<0,$$

or equivalently,

$$\begin{bmatrix} \mathcal{K} & I \end{bmatrix}\begin{bmatrix} I+\mathcal{H}\mathcal{H}^* & -\mathcal{H}\mathcal{L}^* \\ -\mathcal{L}\mathcal{H}^* & -\gamma^2 I+\mathcal{L}\mathcal{L}^* \end{bmatrix}\begin{bmatrix} \mathcal{K}^* \\ I \end{bmatrix}<0. \tag{1.4.7}$$

Now it can be shown (we shall provide the proof shortly) that a causal $\mathcal{K}$ that guarantees the inequality (1.4.7) can be found if, and only if, the center block operator (or matrix, in the finite-horizon case) in (1.4.7) admits the following canonical factorization

$$\begin{bmatrix} I+\mathcal{H}\mathcal{H}^* & -\mathcal{H}\mathcal{L}^* \\ -\mathcal{L}\mathcal{H}^* & -\gamma^2 I+\mathcal{L}\mathcal{L}^* \end{bmatrix}=\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix}, \tag{1.4.8}$$

where $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ is causal and causally invertible, and, in addition, $\mathcal{L}_{11}$ is causal and causally invertible and $\mathcal{L}_{12}$ is strictly causal. In the matrix case, this means that $\mathcal{L}_{11}$, $\mathcal{L}_{21}$ and $\mathcal{L}_{22}$ are lower triangular and that $\mathcal{L}_{12}$ is strictly lower triangular. When the $\mathcal{L}_{ij}$ can be represented by transfer matrices, this means that $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ is minimum phase and proper, and, in addition, that $L_{11}(z)$ is minimum phase and proper and $L_{12}(z)$ is stable and strictly proper.

In practice, the factorization (1.4.8) can be achieved in various ways. In the matrix case it can be obtained via the Krein space Kalman filter (a generalization of the classical Kalman filter to indefinite metric spaces – see Chapter 2), and in the transfer matrix case it can be obtained via J-spectral factorization [KS94].

Once the factorization (1.4.8) has been performed we may rewrite (1.4.7) as follows

$$(\mathcal{K}\mathcal{L}_{11}+\mathcal{L}_{21})(\mathcal{K}\mathcal{L}_{11}+\mathcal{L}_{21})^*-(\mathcal{K}\mathcal{L}_{12}+\mathcal{L}_{22})(\mathcal{K}\mathcal{L}_{12}+\mathcal{L}_{22})^*<0. \tag{1.4.9}$$

Since both $(\mathcal{K}\mathcal{L}_{11}+\mathcal{L}_{21})$ and $(\mathcal{K}\mathcal{L}_{12}+\mathcal{L}_{22})$ are causal, this implies that we must have

$$(\mathcal{K}\mathcal{L}_{11}+\mathcal{L}_{21})=(\mathcal{K}\mathcal{L}_{12}+\mathcal{L}_{22})\mathcal{Q}, \tag{1.4.10}$$

for some *causal strictly contractive* $\mathcal{Q}$ (*i.e.* , a causal $\mathcal{Q}$ such that $\mathcal{Q}\mathcal{Q}^*<I$).

We can now solve the above equation for $\mathcal{K}$ and obtain the following result.

Theorem 1.4.1 (H^∞ **Suboptimal Estimators)** *A causal estimator, $\mathcal{K}$ that achieves*

$$\left\| \begin{bmatrix} \mathcal{L}-\mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix} \right\|_\infty < \gamma,$$

exists if, and only if, there exists a canonical factorization of the form

$$\begin{bmatrix} I+\mathcal{H}\mathcal{H}^* & -\mathcal{H}\mathcal{L}^* \\ -\mathcal{L}\mathcal{H}^* & -\gamma^2 I+\mathcal{L}\mathcal{L}^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix},$$

with $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ *and $\mathcal{L}_{11}$ causal and causally invertible, and $\mathcal{L}_{12}$ strictly causal. If this is the case, then all possible H^∞ estimators of level γ are given by*

$$\mathcal{K} = (\mathcal{L}_{22}\mathcal{Q}-\mathcal{L}_{21})(\mathcal{L}_{11}-\mathcal{L}_{12}\mathcal{Q})^{-1}, \tag{1.4.11}$$

where $\mathcal{Q}$ is any causal and strictly contractive operator. An important choice results from taking $\mathcal{Q}=0$, so that

$$\mathcal{K}_{cen} = -\mathcal{L}_{21}\mathcal{L}_{11}^{-1}, \tag{1.4.12}$$

which is the so-called "central" filter.

Proof: We have practically shown sufficiency already. Solving for $\mathcal{K}$ in (1.4.10), yields (1.4.11). What of course remains to be shown is that for any causal contractive $\mathcal{Q}$ the operator, $\mathcal{K}$, is causal.[11] To show this, let us write

$$\mathcal{K} = (\mathcal{L}_{22}\mathcal{Q}-\mathcal{L}_{21})\mathcal{L}_{11}(I-\mathcal{L}_{11}^{-1}\mathcal{L}_{12}\mathcal{Q})^{-1}, \tag{1.4.13}$$

and note that the causal operator $\mathcal{A}=\mathcal{L}_{11}^{-1}\mathcal{L}_{12}\mathcal{Q}$ is strictly contractive since,

$$\begin{aligned} \mathcal{A}\mathcal{A}^* &= \mathcal{L}_{11}^{-1}\mathcal{L}_{12}\mathcal{Q}\mathcal{Q}^*\mathcal{L}_{12}^*\mathcal{L}_{11}^{-*} \\ &< \mathcal{L}_{11}^{-1}\mathcal{L}_{12}\mathcal{L}_{12}^*\mathcal{L}_{11}^{-*}, && \text{since } \mathcal{Q}\mathcal{Q}^*<I \\ &= \mathcal{L}_{11}^{-1}(\mathcal{L}_{11}\mathcal{L}_{11}^*-I-\mathcal{H}\mathcal{H}^*)\mathcal{L}_{11}^{-*}, && \text{equating the } (1,1) \text{ entries in (1.4.8)} \\ &< \mathcal{L}_{11}^{-1}\mathcal{L}_{11}\mathcal{L}_{11}^*\mathcal{L}_{11}^{-*} \\ &= I. \end{aligned}$$

[11] We could lose the causality of $\mathcal{K}$ if $\mathcal{L}_{11}-\mathcal{L}_{12}\mathcal{Q}$ is not causally invertible.

This means that we can expand the inverse in (1.4.13) to write

$$\mathcal{K} = (\mathcal{L}_{22}\mathcal{Q} - \mathcal{L}_{21})\mathcal{L}_{11}(I + \mathcal{A} + \mathcal{A}^2 + \ldots), \tag{1.4.14}$$

where the infinite series converges absolutely since $\mathcal{A}$ is a strict contraction (see [Hir62], [GK69], [SNF70]). Finally, $\mathcal{K}$ is causal since all the operators appearing in (1.4.14) are causal.

The proof of necessity is slightly more detailed. The proof presented here is of independent interest since it also establishes the maximum entropy property of the central solution, (1.4.12), (see also [GM89] and [MG90]).

To this end, suppose that a causal $\mathcal{K}$ exists that solves the H^∞ estimation problem of level γ. In other words, there exists a causal $\mathcal{K}$ such that (1.4.7) is satisfied, which after a completion of squares argument can be written as

$$\begin{gathered}\gamma^2 I - \mathcal{L}(I + \mathcal{H}^*\mathcal{H})^{-1}\mathcal{L}^* - \\ \left(\mathcal{K}(I + \mathcal{H}\mathcal{H}^*)^{1/2} - \mathcal{L}\mathcal{H}^*(I + \mathcal{H}\mathcal{H}^*)^{-*/2}\right)\left(\mathcal{K}(I + \mathcal{H}\mathcal{H}^*)^{1/2} - \mathcal{L}\mathcal{H}^*(I + \mathcal{H}\mathcal{H}^*)^{-*/2}\right)^* > 0.\end{gathered} \tag{1.4.15}$$

Defining, the causal operator, $\mathcal{F} \triangleq \mathcal{K}(I + \mathcal{H}\mathcal{H}^*)^{1/2}$, and the operator $\mathcal{T} \triangleq \mathcal{L}\mathcal{H}^*(I + \mathcal{H}\mathcal{H}^*)^{-*/2}$, we may write the above equation as

$$\gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F} - \mathcal{T})(\mathcal{F} - \mathcal{T})^* > 0. \tag{1.4.16}$$

Consider now cost function

$$\varepsilon_\gamma(\mathcal{F}) = \log\ \det\left[\gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F} - \mathcal{T})(\mathcal{F} - \mathcal{T})^*\right], \tag{1.4.17}$$

subject to the positivity constraint (1.4.16). The above cost function is referred to as the *entropy cost* (due to the similarity of its form to information theoretic entropy[12] [MG90], [CT91]). Note that, over the set (1.4.16), $\varepsilon_\gamma(\mathcal{F})$ is a concave function and that its value approaches negative infinity as $\mathcal{F}$ approaches the boundary of (1.4.16). Therefore as the set defined via (1.4.16) is convex and nonempty (we have assumed that a causal H^∞ estimator of level γ exists), there will exist a unique causal operator, which we denote by $\mathcal{F}_{opt}$, that maximizes $\varepsilon_\gamma(\mathcal{F})$ subject to the constraint (1.4.16) (see

[12]The entropy of a stationary zero-mean Gaussian process with variance, R_x, is given by $\log\det R_x$.

e.g., [Gol72], [NN93], [BGFB94] on convex optimization theory). In other words, there exists a solution to the following convex optimization problem,

$$\begin{cases} \max_{\text{causal } \mathcal{F}} & \varepsilon_\gamma(\mathcal{F}) \\ \text{subject to} & (1.4.16) \end{cases} \tag{1.4.18}$$

Now since $\mathcal{F}_{opt}$ lies inside the constraint set, it is straightforward to see that it should make the derivative of $\varepsilon_\gamma(\mathcal{F})$ with respect to causal $\mathcal{F}$ equal to zero. Computing this derivative shows that $\mathcal{F}_{opt}$ satisfies the equation

$$\left\{\left[\gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F}_{opt} - \mathcal{T})(\mathcal{F}_{opt} - \mathcal{T})^*\right]^{-1}(\mathcal{F}_{opt} - \mathcal{T})\right\}_+ = 0. \tag{1.4.19}$$

To summarize what has been shown so far, if the H^∞ estimation problem has a solution then (1.4.19) has a solution that satisfies the constraint (1.4.16). Solving Eq. (1.4.19) seems a formidable task, but we shall again show that certain factorizations provide the trick.

Due to (1.4.19) we readily see that

$$\gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F}_{opt} - \mathcal{T})(\mathcal{F}_{opt} - \mathcal{T})^* > 0,$$

so that we can introduce the canonical spectral factorization

$$\mathcal{L}_\Delta\mathcal{L}_\Delta^* = \gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F}_{opt} - \mathcal{T})(\mathcal{F}_{opt} - \mathcal{T})^*, \tag{1.4.20}$$

with $\mathcal{L}_\Delta$ causal and causally invertible. With this definition (1.4.19) can be written as

$$\left\{\mathcal{L}_\Delta^{-*}\mathcal{L}_\Delta^{-1}(\mathcal{F}_{opt} - \mathcal{T})\right\}_+ = 0. \tag{1.4.21}$$

Note that the above equation implies that the operator appearing within the $\{\cdot\}_+$ must be strictly anticausal. Since $\mathcal{L}_\Delta^{-*}$ is anticausal, this implies that $\mathcal{L}_\Delta^{-1}(\mathcal{F}_{opt} - \mathcal{T})$ must be strictly anticausal.[13] Denoting this strictly anticausal operator by $-\bar{\mathcal{L}}_{12}^*$, we have

$$\mathcal{F}_{opt} - \mathcal{T} = -\mathcal{L}_\Delta\bar{\mathcal{L}}_{12}^*. \tag{1.4.22}$$

[13] Denote the strictly anticausal operator appearing within the $\{\cdot\}_+$ by $\mathcal{S} \triangleq \mathcal{L}_\Delta^{-*}\mathcal{L}_\Delta^{-1}(\mathcal{F}_{opt} - \mathcal{T})$. Then $\mathcal{L}_\Delta^{-1}(\mathcal{F}_{opt} - \mathcal{T}) = \mathcal{L}_\Delta^*\mathcal{S}$ must be strictly anticausal since it is the product of an anticausal operator (*i.e.*, $\mathcal{L}_\Delta^*$) and a strictly anticausal operator (*i.e.*, $\mathcal{S}$).

Likewise, let us define the causal and causally invertible transfer operator $\bar{\mathcal{L}}_{11}$ via the canonical factorization

$$\bar{\mathcal{L}}_{11}\bar{\mathcal{L}}_{11}^* = I + \bar{\mathcal{L}}_{12}\bar{\mathcal{L}}_{12}^*. \tag{1.4.23}$$

Finally, let us define the causal operators $\mathcal{L}_{21}$ and $\mathcal{L}_{22}$ via

$$\mathcal{L}_{21} = -\mathcal{F}_{opt}\bar{\mathcal{L}}_{11} \quad \text{and} \quad \mathcal{L}_{22} = \mathcal{L}_\Delta + \mathcal{L}_{21}\bar{\mathcal{L}}_{11}^{-1}\bar{\mathcal{L}}_{12}. \tag{1.4.24}$$

With the above two definitions it is not hard to show that

$$-\mathcal{T} = \mathcal{L}_{21}\bar{\mathcal{L}}_{11}^* - \mathcal{L}_{22}\bar{\mathcal{L}}_{12}^*, \tag{1.4.25}$$

and

$$-\gamma^2 I + \mathcal{L}\mathcal{L}^* = \mathcal{L}_{21}\mathcal{L}_{21}^* - \mathcal{L}_{22}\mathcal{L}_{22}^*. \tag{1.4.26}$$

Combining (1.4.23), (1.4.25) and (1.4.26) shows that the causal and causally invertible $\bar{\mathcal{L}}_{11}$, the strictly causal $\bar{\mathcal{L}}_{12}$, and the causal $\mathcal{L}_{21}$ and $\mathcal{L}_{22}$ can be found from the factorization

$$\begin{bmatrix} I & -\mathcal{T}^* \\ -\mathcal{T} & -\gamma^2 I + \mathcal{L}\mathcal{L}^* \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{L}}_{11} & \bar{\mathcal{L}}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \bar{\mathcal{L}}_{11}^* & \mathcal{L}_{21}^* \\ \bar{\mathcal{L}}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix}. \tag{1.4.27}$$

Note that, from the definition (1.4.24), we can readily find the desired solution to Eq. (1.4.19) using the entries of the above factorization via

$$\mathcal{F}_{opt} = -\mathcal{L}_{21}\bar{\mathcal{L}}_{11}^{-1}. \tag{1.4.28}$$

Finally defining $\mathcal{L}_{11} \triangleq \bar{\mathcal{L}}_{11}(I+\mathcal{H}\mathcal{H}^*)^{1/2}$ and $\mathcal{L}_{12} \triangleq \bar{\mathcal{L}}_{12}(I+\mathcal{H}\mathcal{H}^*)^{1/2}$ and using the definition of $\mathcal{T}$ shows that we have the following factorization

$$\begin{bmatrix} I+\mathcal{H}\mathcal{H}^* & -\mathcal{H}\mathcal{L}^* \\ -\mathcal{L}\mathcal{H}^* & -\gamma^2 I + \mathcal{L}\mathcal{L}^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix},$$

with $\mathcal{L}_{11}$ causal and causally invertible, $\mathcal{L}_{21}$ and $\mathcal{L}_{22}$ causal, and $\mathcal{L}_{12}$ strictly causal. Therefore all that remains to show is that $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ is causally invertible. But this readily follows from the block triangular factorization

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathcal{L}_{21}\mathcal{L}_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11} & 0 \\ 0 & \underbrace{\mathcal{L}_{22} - \mathcal{L}_{21}\mathcal{L}_{11}^{-1}\mathcal{L}_{12}}_{\mathcal{L}_\Delta} \end{bmatrix} \begin{bmatrix} I & \mathcal{L}_{11}^{-1}\mathcal{L}_{12} \\ 0 & I \end{bmatrix}$$

and the facts that $\mathcal{L}_{11}$ and $\mathcal{L}_\Delta$ are causally invertible.

$\blacksquare$

Note that the above proof establishes the following Corollary.

Corollary 1.4.1 (Maximum Entropy Property of Central Solution) *The solution to the problem*

$$\max_{\text{causal } \mathcal{K}} \ \log \det \left[\gamma^2 I - \mathcal{T}_\mathcal{K} \mathcal{T}_\mathcal{K}^*\right] \tag{1.4.29}$$

is given by the central estimator

$$\mathcal{K}_{cen} = -\mathcal{L}_{21}\mathcal{L}_{11}^{-1} \tag{1.4.30}$$

Proof: The proof follows from the facts that

$$\gamma^2 I - \mathcal{T}_\mathcal{K}\mathcal{T}_\mathcal{K}^* = \gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F} - \mathcal{T})(\mathcal{F} - \mathcal{T})^*,$$

so that the cost function in (1.4.29) is simply $\varepsilon_\gamma(\mathcal{F})$, and that $\mathcal{K} = \mathcal{F}(I + \mathcal{H}\mathcal{H}^*)^{-1/2}$ and

$$\mathcal{F}_{opt} = -\mathcal{L}_{21}\bar{\mathcal{L}}_{11}^{-1} = -\mathcal{L}_{21}\mathcal{L}_{11}^{-1}(I + \mathcal{H}\mathcal{H}^*)^{1/2}.$$

$\blacksquare$

Remark: The major conclusion to be made from above discussions is that the solution to the H^∞ estimation problem can be obtained by a suitable canonical spectral factorization of an *indefinite* transfer operator. In this sense, the H^∞ solution is a certain generalization of the H^2 solution. We will have much more to say about this in the next chapter.

1.4.2 Special Cases

The general solution presented in the previous section subsumes as special cases the finite and infinite-horizon H^∞ estimation problems and the H^∞ adaptive filtering problem. We shall now briefly present these.

The Infinite Horizon Case

In the infinite-horizon time-invariant case the relationship between the measurement and desired signals is given by the transfer matrices

$$\left\{ \begin{array}{rcl} y(z) & = & H(z)u(z) + v(z) \\ s(z) & = & L(z)u(z) \end{array} \right. . \tag{1.4.31}$$

In particular, when $H(z)$ and $L(z)$ have finite-dimensional state-space descriptions, we may write

$$\left\{ \begin{array}{rcl} H(z) & = & H(zI-F)^{-1}G \\ L(z) & = & L(zI-F)^{-1}G \end{array} \right. . \tag{1.4.32}$$

Under the above assumptions, Theorem 1.4.1 takes the following form.

Theorem 1.4.2 (Infinite Horizon H^∞ Suboptimal Estimators) *(a) A causal estimator, $K(z)$, that achieves*

$$\left\| \begin{bmatrix} L(z) - K(z)H(z) & -K(z) \end{bmatrix} \right\|_\infty < \gamma,$$

exists if, and only if, there exists a factorization of the form

$$\begin{gathered} \begin{bmatrix} I_p + H(z)H^*(z^{-*}) & -H(z)L^*(z^{-*}) \\ -L(z)H^*(z^{-*}) & -\gamma^2 I_q + L(z)L^*(z^{-*}) \end{bmatrix} = \\ \begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \begin{bmatrix} L_{11}^*(z^{-*}) & L_{21}^*(z^{-*}) \\ L_{12}^*(z^{-*}) & L_{22}^*(z^{-*}) \end{bmatrix}, \end{gathered} \tag{1.4.33}$$

with $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ *and $L_{11}(z)$ minimum phase and proper, and $L_{12}(z)$ causal and strictly proper. If this is the case, then all possible H^∞ estimators of level γ are given by*

$$K(z) = \left(L_{22}(z)Q(z) - L_{21}(z)\right)\left(L_{11}(z) - L_{12}(z)Q(z)\right)^{-1}, \tag{1.4.34}$$

where $Q(z)$ is any causal and strictly contractive transfer matrix, i.e., *$Q(e^{j\omega})Q^*(e^{j\omega}) < I$, $\forall \omega \in [0, 2\pi]$. The central filter results from the choice $Q(z) = 0$, so that*

$$K_{cen}(z) = -L_{21}(z)L_{11}^{-1}(z). \tag{1.4.35}$$

(b) Assume that $H(z)$ and $L(z)$ have state-space structure,

$$\left\{ \begin{array}{rcl} H(z) & = & H(zI-F)^{-1}G \\ L(z) & = & L(zI-F)^{-1}G \end{array} \right. ,$$

with $\{F,G\}$ stabilizable and $\{F,H\}$ detectable. Then a solution to the H^∞ estimation problem with level γ exists if, and only if, there exists a solution to the DARE

$$P = FPF^* + GG^* - K_p R_e K_p^*, \tag{1.4.36}$$

with

$$K_p = FP \begin{bmatrix} H^* & L^* \end{bmatrix} R_e^{-1} \quad \text{and} \quad R_e = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} H \\ L \end{bmatrix} P \begin{bmatrix} H^* & L^* \end{bmatrix} \tag{1.4.37}$$

such that

(i) $F_p \triangleq F - K_p \begin{bmatrix} H \\ L \end{bmatrix}$ *is stable.*

(ii) R_e *and* $\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}$ *have the same inertia*[14].

If this is the case, then the $L_{ij}(z)$ in the canonical factorization (1.4.33) are given by

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \left(\begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} + \begin{bmatrix} H \\ -L \end{bmatrix} (zI-F)^{-1} K_p \right) \times \tag{1.4.38}$$
$$\begin{bmatrix} (I_p + HPH^*)^{1/2} & 0 \\ -LPH^*(I_p+HPH^*)^{-*/2} & (\gamma^2 I_q - LP(I+H^*HP)^{-1}L^*)^{1/2} \end{bmatrix}.$$

[14]By the inertia of a Hermitian matrix, we mean the number of its positive, negative and zero eigenvalues. A simple way of calculating the inertia of a strongly regular Hermitian matrix R (*i.e.* one whose leading minors are all nonzero), is by computing its LDU decomposition

$$R = LDL^*,$$

where L is a lower triangular matrix with unit diagonal and D is a diagonal matrix: the number of positive and negative elements of D give the number of positive and negative eigenvalues of R and hence the inertia.

In particular, defining $R_{He} \triangleq I_p + HPH^*$, *we have*

$$\left\{ \begin{array}{lcl} L_{11}(z) & = & \left[I_p + H(zI - F)^{-1}FPH^*R_{He}^{-1}\right] R_{He}^{1/2} \\ L_{21}(z) & = & -\left[LPH^*R_{He}^{-1} + L(zI - F)^{-1}FPH^*R_{He}^{-1}\right] R_{He}^{1/2} \end{array} \right. , \tag{1.4.39}$$

so that

$$K_{cen}(z) = LPH^*R_{He}^{-1} + L\left(I - PH^*R_{He}^{-1}H\right)(zI - F_1)^{-1}FPH^*R_{He}^{-1}, \tag{1.4.40}$$

where $F_1 \triangleq F - FPH^*R_{He}^{-1}H$.

Proof: Part (a) is simply a restatement of Theorem 1.4.1 when we have transfer matrices.

In part (b), the relation between canonical spectral factorizations of rational matrix functions and the solution to certain algebraic Riccati equations is wellknown (see *e.g.*, [BLW91], [LR95]) and will be studied in detail in Chapter 7. There we will prove that a canonical factorization of the desired form exists if, and only if, a solution to the DARE (1.4.36) with the aforementioned properties exists.

Here we shall confine ourselves to showing that if a solution to the DARE with properties (i) and (ii) exists, then the $L_{ij}(z)$ have all the required properties. To this end, first note that from (1.4.38) it follows that the $L_{ij}(z)$ are stable (since F is stable[15]) and in particular, that $L_{12}(z)$ is strictly proper. Moreover,

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}^{-1} \propto \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} H \\ -L \end{bmatrix} (zI - F_p)^{-1}K_p,$$

where the notation '$\propto$ means that the two matrices are related by a constant invertible matrix. Since F_p is stable the causality of $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}^{-1}$ is established. Finally, note that

$$L_{11}(z)^{-1} \propto I_p - H(zI - F + K_1H)^{-1}K_1,$$

where we have defined $K_1 = FPH^*(I_p + HPH^*)^{-1}$. Therefore to show the causality of $L_{11}(z)^{-1}$ we need to demonstrate that $F - K_1H$ is stable. To do so, we proceed

[15] We have actually assumed instead that $\{F, H\}$ is detectable. In Chapter 7 we shall show that there is no loss in making this weaker assumption.

as follows. First, in Chapter 7 we shall show that if P is a stabilizing solution to the DARE (1.4.36) then $P \geq 0$. This then implies that $I_p + HPH^* > 0$, so that for the inertia condition (ii) to hold the Schur complement of R_e with respect to its $(1,1)$ block entry must be negative definite, *i.e.*,

$$-\gamma^2 I_q + LPL^* - LPH^*(I_p + HPH^*)^{-1}HPL^* = -\gamma^2 I_q + LP(I + H^*HP)^{-1}L^* < 0.$$

Now we can rewrite the DARE (1.4.36) as

$$P = (F - K_1H)P(F - K_1H)^* + GG^* + K_1K_1^* + \underbrace{FP(I + H^*HP)^{-1}L^*\left(\gamma^2 I_q - LP(I + H^*HP)^{-1}L^*\right)^{-1} L(I + PH^*H)^{-1}PF^*}_{\geq 0} .$$

Now since $P \geq 0$ and $\{F - K_1H, \begin{bmatrix} G & K_1 \end{bmatrix}\}$ is stabilizable (since $\{F, G\}$ is stabilizable), Lyapunov theory proves that $F - K_1H$ is stable[16] [Lya67], [GQ95].

■

Remarks:

(i) Despite the fundamental differences in their approaches, the above H^∞ filter and the Wiener filter of Theorem 1.3.2 bear striking resemblances. Indeed the only difference between the central H^∞ filter (1.4.40) and the Wiener filter (1.3.28) is that the Riccati variables, P, satisfy different DARE's. We will have more to say about this in the next subsection and in Chapter 3.

(ii) It is straightforward to give state-space representations for the central filter (1.4.40). Indeed we have

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1} & = & (F - K_1H)\hat{x}_i + K_1 y_i \\ \hat{s}_{i|i} & = & L(I - PH^*R_{He}^{-1}H)\hat{x}_i + LPH^*R_{He}^{-1}y_i \end{array} \right. , \tag{1.4.41}$$

with $K_1 = FPH^*R_{He}^{-1}$ and $R_{He} = I_p + HPH^*$, which is referred to as the predicted form of the central estimator. Defining $\hat{x}_{i|i} \triangleq \hat{x}_i + PH^*R_{He}^{-1}(y_i - H\hat{x}_i)$, we can write the following so-called filtered form of the central estimator

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1|i+1} & = & F\hat{x}_{i|i} + PH^*R_{He}^{-1}(y_{i+1} - HF\hat{x}_{i|i}) \\ \hat{s}_{i|i} & = & L\hat{x}_{i|i} \end{array} \right. . \tag{1.4.42}$$

[16]According to a theorem in stability theory, if the Lyapunov equation $P = FPF^* + Q$, where $Q \geq 0$ and $\{F, Q^{1/2}\}$ is stabilizable, has a solution, $P \geq 0$, then F is a stable matrix.

The Finite Horizon Case

When the measurement and observations processes have a (possibly time-variant) state-space model, we may write

$$\left\{\begin{array}{rcl} x_{i+1} & = & F_i x_i + G_i u_i, \quad N \geq i \geq 0, \ x_0 \\ y_i & = & H_i x_i + v_i \\ s_i & = & L_i x_i \end{array}\right. . \tag{1.4.43}$$

In this case, the disturbances are expanded to include the unknown initial state, x_0, so that the transfer operator in question becomes,

$$\mathcal{T}_{\mathcal{K}}: \ \left\{\Pi_0^{-1/2} x_0, \{u_i\}_{i=0}^N, \{v_i\}_{i=0}^N\right\} \rightarrow \{s_i - \hat{s}_{i|i}\}_{i=0}^N. \tag{1.4.44}$$

where $\Pi_0 > 0$ reflects the weight that is given to uncertainty in the initial condition relative to uncertainty in the u_i's and v_i's. Therefore the finite-horizon suboptimal H^∞ estimation problem can be formulated as follows.

Problem 1.4.3 (Sub-optimal Finite Horizon H^∞ Estimation Problem) *Given a scalar $\gamma > 0$ and a final time, N, find an H^∞ sub-optimal estimator $\hat{s}_{j|j} = \mathcal{K}(y_0, \ldots, y_j)$ that achieves $\|\mathcal{T}_{\mathcal{K}}\|_\infty < \gamma$. In other words,*

$$\sup_{x_0, u \in h_2, v \in h_2} \frac{\sum_{j=0}^N \tilde{s}_{j|j}^* \tilde{s}_{j|j}}{x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^N u_j^* u_j + \sum_{j=0}^N v_j^* v_j} < \gamma^2. \tag{1.4.45}$$

This clearly requires checking whether $\gamma > \gamma_{opt}$.

The solution to the above problem is given below (see [KN91], [ST92], [HSK96b], [HSK93b], [GL95]).

Theorem 1.4.3 (Finite Horizon H^∞ Filter) *An H^∞ filter of level γ exists if, and only if, the matrices*

$$R_i = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} \ \text{and} \ R_{e,i} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} H_i \\ L_i \end{bmatrix} P_i \begin{bmatrix} H_i^* & L_i^* \end{bmatrix} \tag{1.4.46}$$

have the same inertia for all $0 \leq i \leq N$, where $P_0 = \Pi_0$ and P_i satisfies the Riccati recursion

$$P_{i+1} = F_i P_i F_i^* + G_i G_i^* - F_i P_i \begin{bmatrix} H_i^* & L_i^* \end{bmatrix} R_{e,i}^{-1} \begin{bmatrix} H_i \\ L_i \end{bmatrix} P_i F_i^*. \tag{1.4.47}$$

If this is the case, then the central H^∞ estimator is given by

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1} & = & F_i \hat{x}_i + K_{1,i}(y_i - H_i \hat{x}_i), \quad \hat{x}_0 = 0 \\ \hat{s}_{i|i} & = & L_i \hat{x}_i + L_i P_i H_i^* R_{He,i}^{-1}(y_i - H_i \hat{x}_i) \end{array} \right. , \tag{1.4.48}$$

with $K_{1,i} = F_i P_i H_i^* R_{He,i}^{-1}$ and $R_{He,i} = I_p + H_i P_i H_i^*$. If we define $\hat{x}_{i|i} \triangleq \hat{x}_i + P_i H_i^* R_{He,i}^{-1}(y_i - H_i \hat{x}_i)$, we can alternatively write the central estimator as

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1|i+1} & = & F_i \hat{x}_{i|i} + P_i H_i^* R_{He,i}^{-1}(y_{i+1} - H_{i+1} F_i \hat{x}_{i|i}), \quad \hat{x}_{-1|-1} = 0 \\ \hat{s}_{i|i} & = & L_i \hat{x}_{i|i} \end{array} \right. . \tag{1.4.49}$$

Proof: The proof will be given in Chapter 3.

∎

Note that the filter of Theorem 1.4.3 looks very much like a Kalman filter solution, except that the Riccati recursion differs from that of the Kalman filter, since

(i) we have indefinite *covariance* matrices, $\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}$.

(ii) the L_i (of the quantity to be estimated) enters the Riccati equation.

(iii) we have an additional condition, (1.4.46), that must be satisfied for the filter to exist; in the Kalman filter problem the L_i would not appear, and the P_i would be positive definite, so that (1.4.46) is immediate.

The appearance of the L_i means that the H^∞ estimate of say, the first component of the state vector x_i, is not the first component of the H^∞ estimate of the whole state vector (because in the first case $L_i = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$, and in the second case $L_i = I$). This is very different from the situation in the H^2 case, where the estimate of any linear combination of the state is simply given by that linear combination of the state estimate.

Despite these differences, we will see in Chapter 2 (see also [HSK96c], [HSK96b], [HSK93c],[HSK93b]) that the filter of Theorem 1.4.3 can in fact be obtained as a certain Kalman filter, not in an H^2 (Hilbert) space, but in a certain indefinite metric space, called a Krein space. The indefinite covariances and the appearance of L_i in the Riccati equation are all easily explained in this framework. The additional condition (1.4.46) will be seen to arise from the fact that in Krein space, unlike as in the usual Hilbert space context, quadratic forms need not always have minima or maxima, unless certain additional conditions are met.

We finally mention that when the system matrices are constant, under some mild conditions, the above finite-horizon H^∞ filter converges to the infinite-horizon H^∞ of Theorem 1.4.2, as time progresses to infinity. The asymptotic behaviour of H^∞ and H^2 filters will be dealt with in Chapter 8.

The LMS Algorithm

As noted in Sec. 1.2.1, the adaptive filtering problem can be recast as a special case of a state-space estimation problem where the state-space model has the form,

$$\left\{ \begin{array}{rcl} x_{i+1} & = & x_i \\ d_i & = & h_i^T x_i + v_i \\ s_i & = & L_i x_i \end{array} \right. , \quad i \geq 0, \quad x_0 = w. \tag{1.4.50}$$

i.e., $F_i = I$, $G_i = 0$ and $H_i = h_i^T$. Therefore to obtain H^∞ adaptive filters all one needs to do is to apply the above state-space model to the filter of Theorem 1.4.3. However, since in adaptive filtering one does not usually know the input vector $\{h_i\}$ beforehand, it is useful to have some method of obtaining a priori bounds on the value of admissible γ's. This problem will be dealt with in Chapter 11 (see also [HK94a], [HK95a]).

However, in the important special case of estimating the uncorrupted output, *i.e.*, when $L_i = h_i^T$, it turns out that one can solve the optimal H^∞ adaptive filtering problem! It so happens that in this case, $\gamma_{opt} = 1$, *i.e.*, there is no amplification of the disturbances. Moreover, the central H^∞-optimal adaptive algorithm is the celebrated normalized *least-mean-squares* (LMS) algorithm of Widrow and Hoff [WH60],

$$\hat{w}_{|i} = \hat{w}_{|i-1} + \frac{h_i}{\frac{1}{\mu} + h_i^T h_i}(d_i - h_i^T \hat{w}_{|i-1}), \quad \hat{w}_{|-1} = 0. \tag{1.4.51}$$

This result, for the first time, furnished a rigorous basis for the LMS algorithm, which was long regarded as an approximate least-squares solution, and provided theoretical justification for its widely observed robust performance (see [HSK96a], [HSK93a], [HSK95]). We will treat these topics in depth in Chapter 9.

1.5 Control Problems

We will now focus our attention on control problems. As mentioned earlier, the main goal in control theory is to influence the dynamical behaviour of a given plant, through a control signal, such that the plant yields some desired performance. For pedagogical reasons it will be more convenient to present two different control problems: first, the full information control problem, where the control signal has access to all the information available up to the current time, and second, the measurement feedback control problem, where the control signal has only access to a certain measurement signal. Although the latter scenario is much more realistic of practical control problems, we shall begin by studying the former problem for two important reasons: first, the full information control problem turns out to be the *dual* of the estimation problem considered so far, and second, the solution to the measurement feedback control problem is simply obtained by estimating (from the measurements) the full information control signal. [This fact is known as the *separation principle* or as the *certainty equivalence principle* in the literature. The first complete proof of this principle in the H^2 case is attributed to Wonham [Won68b] and similar results in H^∞ and risk-sensitive control have been obtained in [DGKF89] and [Whi90].]

1.5.1 Full Information Control

A general discrete-time full information control problem is shown in Fig. 1.3 which subsumes both the finite and infinite horizon cases. Here we assume that $\mathcal{P}_1$ and $\mathcal{P}_2$

are known *causal* linear transfer operators that map the input sequences $w \triangleq \{w_i\}$ and $u \triangleq \{u_i\}$ to the output sequence $s \triangleq \{s_i\}$, *i.e.*,

$$s = \mathcal{P}_1 w + \mathcal{P}_2 u. \tag{1.5.1}$$

As before, in the finite horizon case the $\mathcal{P}_i$ can be represented by lower triangular matrices, and in the infinite horizon time-invariant case they can be represented by transfer matrices, $P_i(z)$.

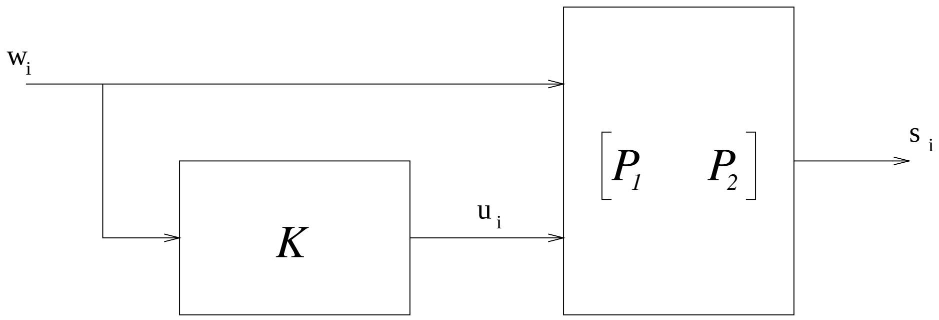

Figure 1.3: The full information control problem.

The signal w is referred to as the *exogenous input*. It can essentially be considered as process noise (or the driving disturbance), and in the finite horizon case it includes the initial state of the plant, x_0. The signal u is referred to as the *control input* and is used to influence the dynamical behaviour of the plant, $\mathcal{P} = \begin{bmatrix} \mathcal{P}_1 & \mathcal{P}_2 \end{bmatrix}$. In the full information case, the control signal at a given time, i, is allowed to be a function of all the information available up to time i, *i.e.*, it is allowed to be a function of $\{w_j, j \leq i\}$ (since all prior information is in the exogenous input, w). This means that we can write

$$u = \mathcal{K} w, \tag{1.5.2}$$

where $\mathcal{K}$ is a causal linear operator referred to as the *controller*. Finally, the signal s is referred to as the *regulated output*, and in the finite horizon case it includes the final state of the plant, x_{N+1}. Roughly speaking, the goal of control is to keep the regulated signal as 'small' as possible (where, of course, 'small' is meant in a certain sense).

It is of course quite conceivable that if the choice of u were not constrained in any way, then it should be possible to make the regulated signal s arbitrarily small. As this may (and typically will) require an arbitrarily large control signal, in order to guarantee the cost-effectiveness of the final control strategy, it is necessary to try to keep the control signal small as well. Therefore we are left with the twofold objective of designing a causal controller, $\mathcal{K}$, that simultaneously guarantees that the regulated signal, s, and the control signal, u, be small. (These are obviously two competing objectives.)

In view of the above remarks, we now note that (roughly speaking) the behaviour of any controller $\mathcal{K}$ can be captured by, $\mathcal{T}^c_{\mathcal{K}}$, the transfer operator that maps the exogenous input, w, to the regulated and control signals, $\{s, u\}$. Thus

$$\mathcal{T}^c_{\mathcal{K}} : \; w \rightarrow \begin{bmatrix} s \\ u \end{bmatrix}. \tag{1.5.3}$$

From Fig. 1.3 it is straightforward to infer that

$$\mathcal{T}^c_{\mathcal{K}} = \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix}. \tag{1.5.4}$$

Remark: Note that if we take the adjoint (conjugate transpose, for matrices) of the operator, $\mathcal{T}^c_{\mathcal{K}}$, we may write

$$\mathcal{T}^{c*}_{\mathcal{K}} = \begin{bmatrix} \mathcal{P}_1^* + \mathcal{K}^*\mathcal{P}_2^* & \mathcal{K}^* \end{bmatrix}. \tag{1.5.5}$$

Comparing the above transfer operator with $\mathcal{T}_{\mathcal{K}}$, the transfer operator under consideration in estimation problems, and given by Eq. (1.2.3), we see that if we make the transformations

$$\mathcal{P}_1^* \leftrightarrow \mathcal{L} \quad , \quad \mathcal{P}_2^* \leftrightarrow \mathcal{H} \quad , \quad \mathcal{K}^* \leftrightarrow -\mathcal{K}$$

then we can obtain one transfer operator from the other. In other words, by replacing causal operators ($\mathcal{L}$ and $\mathcal{H}$) with anticausal operators ($\mathcal{P}_1^*$ and $\mathcal{P}_2^*$) and by insisting on an anticausal estimator ($-\mathcal{K}^*$ instead of $\mathcal{K}$) we see that we can solve the full information control problem by solving a related (so-called *dual*) estimation problem. The above observation is at the heart of the duality between estimation and (full information) control and will be used (to an extent) in the remainder of this chapter to solve the various control problems under consideration. [We shall return to duality in Chapter 6 and study it from a deeper geometric viewpoint. See also the recent paper [Lue92] and the book [Wal89] on duality in mathematical programming.]

1.5.2 Measurement Feedback Control

A general discrete-time measurement feedback control problem is shown in Fig. 1.4 which, as before, subsumes both the finite and infinite horizon cases. Here the $\mathcal{P}_{ij}$ are known causal linear transfer operators that map the input sequences $w \triangleq \{w_i\}$ and $u \triangleq \{u_i\}$ to the output sequences $s \triangleq \{s_i\}$ and $t \triangleq \{t_i\}$ according to the formula

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}. \tag{1.5.6}$$

In the finite horizon case the $\mathcal{P}_{ij}$ can be represented by lower triangular matrices, and in the infinite horizon time-invariant case they can be represented by transfer matrices, $P_{ij}(z)$.

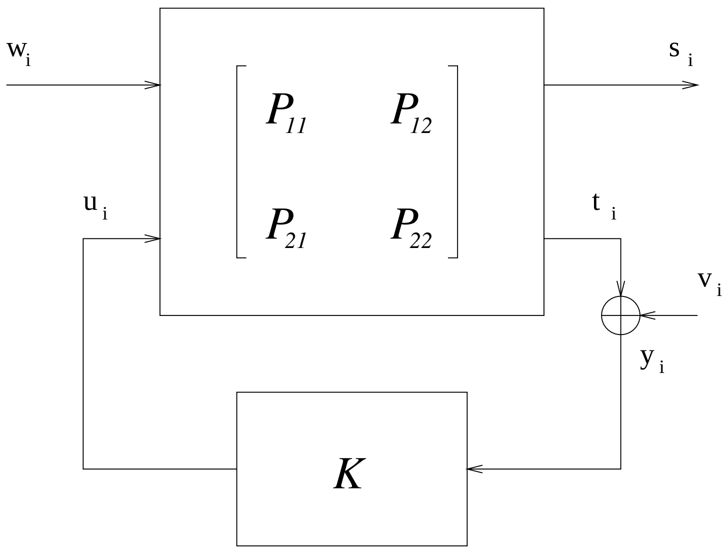

Figure 1.4: The measurement feedback control problem.

Here the sequences w and v are the exogenous signals. The signal w can be considered to be the process noise (or the driving disturbance) and the signal v can be considered to be the measurement noise (which corrupts the output signal, t). The signal u is the control input used to influence the dynamical behaviour of the plant. In the measurement feedback case, the control signal at any given time, i, is only allowed to be a function of current and past observations of y. (The signal y is the so-called measurement signal which can be regarded as a corrupted version of the output signal, t.) In other words, contrary to the full-information case where the control signal had access to all the available information, here u_i is only allowed to be a function of $\{y_j, j \leq i\}$. This means that we can write

$$u = \mathcal{K}y, \tag{1.5.7}$$

for some causal linear operator, $\mathcal{K}$, called the controller. As before, s is the regulated output that we intend to keep small.

In the full information problem we noted that in order to keep the control law cost-effective, it is necessary to ensure that the control signal be not too large. This observation also holds for the measurement feedback problem, so that (as in the previous section) we are confronted with the two-fold task of keeping both the regulated signal, s, as well as the control signal, u, small (in some pre-defined sense).

In view of the above, the behaviour of any controller, $\mathcal{K}$, can be captured by the $\mathcal{T}_{\mathcal{K}}^{c}$, the transfer operator that maps the exogenous inputs, $\{w, v\}$, to the regulated and control signals, $\{s, u\}$, *i.e.*,

$$\mathcal{T}_{\mathcal{K}}^{c} : \begin{bmatrix} w \\ v \end{bmatrix} \to \begin{bmatrix} s \\ u \end{bmatrix}. \tag{1.5.8}$$

Using the model presented so far we can write

$$\begin{cases} s &= \mathcal{P}_{11}w + \mathcal{P}_{12}u \\ y &= \mathcal{P}_{21}w + \mathcal{P}_{22}u + v \\ u &= \mathcal{K}y \end{cases},$$

from which, after some algebra, yields

$$\mathcal{T}_{\mathcal{K}}^{c} = \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix}. \tag{1.5.9}$$

The above expression shows an apparently complicated relationship between the transfer operator $\mathcal{T}_{\mathcal{K}}^c$ and the controller, $\mathcal{K}$. Indeed referring back to Eqs. (1.2.3) and (1.5.4), we see that in the estimation problem and in the full information control problem, the transfer operators, $\mathcal{T}_{\mathcal{K}}$ and $\mathcal{T}_{\mathcal{K}}^c$ were *affine* in the respective estimator and controller, $\mathcal{K}$. Here $\mathcal{T}_{\mathcal{K}}^c$ is obviously not affine in $\mathcal{K}$. However, if we define the transfer operator

$$\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}, \tag{1.5.10}$$

(which is a bilinear transformation of the controller) we immediately see that

$$\mathcal{T}_{\mathcal{K}}^c = \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{Q} \\ \mathcal{Q}\mathcal{P}_{21} & \mathcal{Q} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{P}_{12} \\ I \end{bmatrix} \mathcal{Q} \begin{bmatrix} \mathcal{P}_{21} & I \end{bmatrix}, \tag{1.5.11}$$

which is now affine in $\mathcal{Q}$. The above expression for $\mathcal{T}_{\mathcal{K}}^c$ is the starting point for the Youla-Jabr-Bongiorno-Kucera (YBJK) parametrization of all controllers that stabilize $\mathcal{T}_{\mathcal{K}}^c$, [Kuc74], [Kuc75], [YBJ76a], [YBJ76b]. [Indeed when the original plant is stable, *i.e.*, when the P_{ij} are all stable, it is straightforward to see that $\mathcal{T}_{\mathcal{K}}^c$ is stable if, and only if, $\mathcal{Q}$ is stable. Therefore the set of all stabilizing controllers are those $\mathcal{K}$ that yield a stable $\mathcal{Q}$. The parametrization in the case of an unstable plant is slightly more involved and requires certain interpolation conditions on $\mathcal{Q}$.]

The above formula for $\mathcal{T}_{\mathcal{K}}^c$ is quite useful and may be used for controller design (see *e.g.*, [BB91] and the references therein). We shall make some (although not very explicit) use of it in the solutions to the H^2 and H^∞ measurement feedback control problems. However, for the most part, our solutions will take a somewhat different route.

1.5.3 Special Cases

We now consider some special cases of the above general formulations.

Transfer Matrices

In the infinite-horizon case, when the $\mathcal{P}_i$ (for the full information problem) and the $\mathcal{P}_{ij}$ (for the measurement feedback problem) are linear time-invariant transfer operators, they can be represented by transfer matrices, $P_i(z)$ and $P_{ij}(z)$, respectively. [In particular, given that the w_i, u_i, y_i and s_i are m_1-vectors, m_2-vectors, p-vectors and q-vectors, respectively, the dimensions of the corresponding transfer matrices can be immediately found.] In this case, for the full information problem, we can write

$$\left\{ \begin{array}{rcl} s(z) & = & P_1(z)w(z) + P_2(z)u(z) \\ u(z) & = & K(z)w(z) \end{array} \right. , \tag{1.5.12}$$

and for the measurement feedback problem,

$$\left\{ \begin{array}{rcl} s(z) & = & P_{11}(z)w(z) + P_{12}(z)u(z) \\ y(z) & = & P_{21}(z)w(z) + P_{22}(z)u(z) + v(z) \\ u(z) & = & K(z)y(z) \end{array} \right. . \tag{1.5.13}$$

The transfer operator $\mathcal{T}_{\mathcal{K}}^c$ itself will also have a transfer matrix representation. For the full information problem,

$$T_K^c = \left[\begin{array}{c} P_1(z) + P_2(z)K(z) \\ K(z) \end{array} \right], \tag{1.5.14}$$

and for the measurement feedback problem,

$$T_K^c = \left[\begin{array}{cc} P_{11}(z) + P_{12}(z)K(z)\left(I - P_{22}(z)K(z)\right)^{-1} P_{21}(z) & P_{12}(z)K(z)\left(I - P_{22}(z)K(z)\right)^{-1} \\ K(z)\left(I - P_{22}(z)K(z)\right)^{-1} P_{21}(z) & K(z)\left(I - P_{22}(z)K(z)\right)^{-1} \end{array} \right]. \tag{1.5.15}$$

State-Space Models

For a variety of reasons, it is often convenient to represent the relationship between the exogenous signals, w_i and v_i, the control signal, u_i, the measurement signal, y_i, and the regulated signal, s_i, via a (possibly time-varying) linear state-space model. In this case, for the full information problem we can write, write

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F_i x_i + G_{1,i} w_i + G_{2,1} u_i \\ s_i & = & L_i x_i \end{array} \right. , \tag{1.5.16}$$

and for the measurement feedback problem,

$$\left\{\begin{array}{rcl} x_{i+1} & = & F_i x_i + G_{1,i} w_i + G_{2,1} u_i \\ s_i & = & L_i x_i \\ y_i & = & H_i x_i + v_i \end{array}\right. , \tag{1.5.17}$$

where $F_i \in \mathcal{C}^{n\times n}$, $G_{1,i} \in \mathcal{C}^{n\times m_1}$, $G_{1,2} \in \mathcal{C}^{n\times m_2}$, $H_i \in \mathcal{C}^{p\times n}$ and $L_i \in \mathcal{C}^{q\times n}$ are known system matrices, and where x_i is the n-dimensional state. Note that we have not specified the range of the time index, i, in either of the models (1.5.16) or (1.5.17) since the control problem may be finite, semi-infinite, or infinite horizon. [Note, moreover, that according to (1.5.16) and (1.5.16), since y_i and s_i depend on $\{w_j, j < i\}$ and $\{u_j, j < i\}$, the transfer operators $\mathcal{P}_i$ and $\mathcal{P}_{ij}$ are strictly causal. It turns out that there is no loss of generality in making this assumption. The benefit is that the algebraic expressions obtained are simpler.]

If we assume that the system matrices in (1.5.16) and (1.5.17) are time-invariant, *i.e.*,

$$F_i \triangleq F \quad , \quad G_{1,i} \triangleq G_1 \quad , \quad G_{2,i} \triangleq G_2 \quad , \quad H_i \triangleq H \quad , \quad L_i \triangleq L$$

then in the infinite-horizon case we can readily find the transfer matrices $P_i(z)$ and $P_{ij}(z)$. For the full information problem we have,

$$\left[\begin{array}{cc} P_1(z) & P_2(z) \end{array}\right] = L(zI - F)^{-1}\left[\begin{array}{cc} G_1 & G_2 \end{array}\right], \tag{1.5.18}$$

and for the measurement feedback problem,

$$\left[\begin{array}{cc} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{array}\right] = \left[\begin{array}{c} L \\ H \end{array}\right](zI - F)^{-1}\left[\begin{array}{cc} G_1 & G_2 \end{array}\right]. \tag{1.5.19}$$

1.6 The H^2 Approach

The problem of control (be it full information or measurement feedback) is to select the controller $\mathcal{K}$ so that the transfer operator $\mathcal{T}_{\mathcal{K}}^c$ is small in some sense. This will (roughly speaking) guarantee that the regulated signal, s, and the control signal, u, are simultaneously small. The most widely used criterion for this purpose is the H^2 norm of $\mathcal{T}_{\mathcal{K}}^c$, *i.e.*, $\|\mathcal{T}_{\mathcal{K}}^c\|_2$. The remarks given at the beginning of Sec. 1.3 (on the definition of the H^2 norm in the finite and infinite-horizon cases, and on the reasons for the pervasive use of the H^2 criterion) all apply here as well and will therefore not be repeated.[17] Instead we shall comment on the connections with *linear-quadratic-regulator* (LQR) and *linear-quadratic-Gaussian* (LQG) control.

1.6.1 Connections to LQR and LQG Control

Consider the finite-horizon case and the quadratic cost function,

$$J_N^c = \sum_{i=0}^{N} s_i^* s_i + \sum_{i=0}^{N} u_i^* u_i. \tag{1.6.1}$$

The above cost function is obviously a measure of how large the regulated and control signals are and is an indication of how well our control strategy performs. We can, of course, also write

$$J_N^c = \|s\|_2^2 + \|u\|_2^2. \tag{1.6.2}$$

Assume now that the $\{w_j\}$ and $\{v_j\}$ are zero-mean, uncorrelated and temporally white stochastic processes with unit variance, *i.e.*,

$$E \begin{bmatrix} w_i \\ v_i \end{bmatrix} \begin{bmatrix} w_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} \delta_{ij}. \tag{1.6.3}$$

Then J_N^c itself will be a random variable. Computing the mean value of J_N^c yields, in the full information case,

$$EJ_N^c = E \begin{bmatrix} s^* & u^* \end{bmatrix} \begin{bmatrix} s \\ u \end{bmatrix} = E\ w^* \mathcal{T}_{\mathcal{K}}^{c*} \mathcal{T}_{\mathcal{K}}^{c} w = E\ \text{trace}\left(\mathcal{T}_{\mathcal{K}}^{c*} \mathcal{T}_{\mathcal{K}}^{c} w w^* \right), \tag{1.6.4}$$

and in the measurement feedback case,

$$\begin{aligned} EJ_N^c = E \begin{bmatrix} s^* & u^* \end{bmatrix} \begin{bmatrix} s \\ u \end{bmatrix} &= E \begin{bmatrix} w^* & v^* \end{bmatrix} \mathcal{T}_{\mathcal{K}}^{c*} \mathcal{T}_{\mathcal{K}}^{c} \begin{bmatrix} w \\ v \end{bmatrix} \\ &= E\ \text{trace} \left(\mathcal{T}_{\mathcal{K}}^{c*} \mathcal{T}_{\mathcal{K}}^{c} \begin{bmatrix} w \\ v \end{bmatrix} \begin{bmatrix} w^* & v^* \end{bmatrix} \right). \end{aligned} \tag{1.6.5}$$

[17] An excellent bibliography on developments in H^2 control until 1971 has been complied in [MG71]. See also the 1971 special issue of the IEEE Transactions on Automatic Control devoted to LQG control [Ath71].

In view of (1.6.3), we have $E\begin{bmatrix} w \\ v \end{bmatrix}\begin{bmatrix} w^* & v^* \end{bmatrix} = I$, so that in both the full information and measurement feedback cases the expected quadratic cost functions become,

$$E\left(\sum_{i=0}^{N} s_i^* s_i + \sum_{i=0}^{N} u_i^* u_i\right) = \text{trace}\,(\mathcal{T}_{\mathcal{K}}^{c*}\mathcal{T}_{\mathcal{K}}^{c}) = \|\mathcal{T}_{\mathcal{K}}^{c}\|_2^2, \tag{1.6.6}$$

which is the cost function that H^2-optimal controllers minimize. Therefore, in the finite-horizon case, and under the aforementioned statistical assumptions, H^2-optimal controllers minimize an expected quadratic cost function. This is why they are also referred to as linear least-mean-squares controllers.

Using a similar argument in the infinite-horizon time-invariant case, it is possible to show that

$$E\left(s_i^* s_i + u_i^* u_i\right) = \frac{1}{2\pi}\int_0^{2\pi} \|T_K(e^{j\omega})\|_F^2 d\omega = \|\mathcal{T}_{\mathcal{K}}\|_2^2. \tag{1.6.7}$$

Therefore, in the infinite-horizon case, H^2-optimal controllers also minimize an expected quadratic cost function.

Remarks:

(a) If, in addition to the aforementioned statistical assumptions, the $\{w_j\}$ and $\{v_j\}$ are assumed to be jointly Gaussian, then the H^2-optimal controller is a *least-mean-squares* controller (*i.e.*, we do not need to restrict the controller to being linear). H^2-optimal controllers are therefore often called LQG controllers to emphasize that (L), they are linear, (Q), the cost function is quadratic, and (G), the exogenous inputs are Gaussian.

(b) Consider the full information problem where we have state-space structure. When the driving disturbance is assumed zero, so that the only exogenous variable is the initial state, x_0, it is very easy to see that H^2-optimal controllers solve the following finite and infinite horizon optimization problems,

$$\min_{\{u_i\}} \sum_{i=0}^{N} s_i^* s_i + \sum_{i=0}^{N} u_i^* u_i \quad \text{and} \quad \min_{\{u_i\}} \sum_{i=0}^{\infty} s_i^* s_i + \sum_{i=0}^{\infty} u_i^* u_i. \tag{1.6.8}$$

The above problems are referred to as linear quadratic regulator problems. We shall study them in more detail in Chapter 6.

1.6.2 The Full Information Solution

The solution to the H^2 full information control problem is well known (see *e.g.*, [Bro70], [AM71] and [KS72]) and is given in the following Theorem.

Theorem 1.6.1 (H^2-optimal Full Information Controller) *The solution to the problem*

$$\min_{\text{causal } \mathcal{K}} \left\| \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix} \right\|_2 \tag{1.6.9}$$

is given by

$$\mathcal{K} = -(I + \mathcal{P}_2^*\mathcal{P}_2)^{-1/2} \left\{ (I + \mathcal{P}_2^*\mathcal{P}_2)^{-*/2}\mathcal{P}_2^*\mathcal{P}_1 \right\}_+ \tag{1.6.10}$$

where $(I+\mathcal{P}_2^*\mathcal{P}_2)^{-1/2}$ and $(I+\mathcal{P}_2^*\mathcal{P}_2)^{-*/2}$ are found from the dual canonical (maximum-phase minimum-phase) factorization

$$I + \mathcal{P}_2^*\mathcal{P}_2 = (I + \mathcal{P}_2^*\mathcal{P}_2)^{*/2}(I + \mathcal{P}_2^*\mathcal{P}_2)^{1/2}, \tag{1.6.11}$$

and where the notation $\{\mathcal{A}\}_+$ denotes the causal part of the transfer operator, $\mathcal{A}$.

In (1.6.11) the transfer operator $(I+\mathcal{P}_2^*\mathcal{P}_2)^{1/2}$ is both causal and causally invertible (hence minimum phase) and the transfer operator $(I + \mathcal{P}_2^*\mathcal{P}_2)^{*/2}$ is both anti-causal and anti-causally invertible (hence maximum phase). Note, however, that the factorization (1.6.11) is different from the factorization (1.3.9), required in the solution of the H^2 estimation problem, in the sense that the maximum-phase factor appears on the left and the minimum-phase factor on the right, rather than vice-versa. For this reason, we have called Eq. (1.6.11) the *dual* canonical factorization. It should be remarked that, as with the usual canonical factorization, the dual canonical factorization of a positive-definite operator (such as $I + \mathcal{P}_2^*\mathcal{P}_2$) always exists (see *e.g.* [Spe85],[GK86].) [In the finite-horizon case (1.6.11) is simply the UU^* (block upper-lower triangular) decomposition of the matrix $I + \mathcal{P}_2^*\mathcal{P}_2$.]

As mentioned earlier, the full information control problem can be considered to be the dual of the estimation problem. Although the final solution of Eq. (1.6.10) already shows some of this duality, in order to display the duality in a more revealing fashion let us take the negative conjugate transpose of (1.6.10) to write

$$-\mathcal{K}^* = \left\{ \mathcal{P}_1^*\mathcal{P}_2(I + \mathcal{P}_2^*\mathcal{P}_2)^{-1/2} \right\}_{\text{a.c.}} (I + \mathcal{P}_2^*\mathcal{P}_2)^{*/2}, \tag{1.6.12}$$

where $\{\mathcal{A}\}_{\text{a.c.}}$ denotes the anticausal part of the operator $\mathcal{A}$, and where we have used the readily verified identity $\{\mathcal{A}\}_+^* = \{\mathcal{A}^*\}_{\text{a.c.}}$. Inspection of (1.6.12) reveals that by replacing the anticausal operators $-\mathcal{K}^*$, $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ with the causal operators $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{H}$, replacing $\{\cdot\}_{\text{a.c.}}$ with $\{\cdot\}_+$, and replacing the dual canonical factorization (1.6.11) with the usual factorization, one readily obtains the H^2 estimator of Eq. (1.3.8). Thus the solutions of Theorems 1.3.1 and 1.6.1 are truly the dual of one another.

Proof of Theorem 1.6.1: The proof is the dual to the proof of Theorem 1.3.1 and will not be repeated here.

■

1.6.3 The Measurement Feedback Solution

In the estimation problems of Secs. 1.3 and 1.4 and in the full information control problem considered so far, the estimators and controllers obtained were both causal and stable.[18] This was true even when the underlying transfer operators ($\mathcal{L}$ and $\mathcal{H}$ in estimation, $\mathcal{P}_1$ and $\mathcal{P}_2$ in control) where causal but unstable. The resulting H^2 and H^∞ estimators and full information controllers also lead to causal and stable $\mathcal{T}_\mathcal{K}$ and $\mathcal{T}_\mathcal{K}^c$. However, as we shall presently see, in the measurement feedback problem we can no longer guarantee the stability of the causal controller, $\mathcal{K}$. Despite this, the stability of $\mathcal{T}_\mathcal{K}^c$ can still ge guaranteed.

The solution to the H^2 measurement feedback control problem is given below and, as mentioned earlier, involves a certain two-step procedure known as the separation principle.

[18] Until the present we have deliberately not been very specific about the distinction between causal and stable operators. A causal operator is one for which the mapping from future inputs to past and current outputs is zero, *i.e.*, the output at any given time is only a function of past and current inputs. On the other hand, there are many different notions of stability for linear transfer operators (such as L^2, L^∞ and exponential stability), but they all essentially amount to the fact that if the inputs are bounded (in some sense) then the outputs are also bounded (in some sense). For example, causal operators with transfer matrix representations can formally be written as $H(z) = H_0 + H_1 z^{-1} + \ldots$. However, such operators need not be stable since in such a formal expansion the H_i need not be bounded. L^2 (or H^2) stability, for example, requires that $\sum_j^\infty \text{trace}(H_j^* H_j) < \infty$.

Theorem 1.6.2 (H^2-optimal Measurement Feedback Controller) *The solution to the problem*

$$\min_{\text{causal } \mathcal{K}} \left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix} \right\|_2 \tag{1.6.13}$$

is given by

$$\mathcal{K} = \left((I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2} + \mathcal{K}_e\mathcal{P}_{22} \right)^{-1} \mathcal{K}_e, \tag{1.6.14}$$

where $\mathcal{K}_e = \left\{ (I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f\mathcal{P}_{21}^*(I+\mathcal{P}_{21}\mathcal{P}_{21}^*)^{-*/2} \right\}_+ (I+\mathcal{P}_{21}\mathcal{P}_{21}^*)^{-1/2}$ *is the solution to the estimation problem,*

$$\min_{\text{causal } \mathcal{K}_e} \left\| \begin{bmatrix} (I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f - \mathcal{K}_e\mathcal{P}_{21} & -\mathcal{K}_e \end{bmatrix} \right\|_2, \tag{1.6.15}$$

and $\mathcal{K}_f = -(I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{-1/2} \left\{ (I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{-*/2}\mathcal{P}_{12}^*\mathcal{P}_{11} \right\}_+$ *is the solution to the full information control problem,*

$$\min_{\text{causal } \mathcal{K}_f} \left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}_f \\ \mathcal{K}_f \end{bmatrix} \right\|_2, \tag{1.6.16}$$

with $(I+\mathcal{P}_{21}\mathcal{P}_{21}^*)^{1/2}$ *and* $(I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}$ *given by the canonical and dual canonical factorizations,*

$$\left\{ \begin{array}{rcl} I+\mathcal{P}_{21}\mathcal{P}_{21}^* &=& (I+\mathcal{P}_{21}\mathcal{P}_{21}^*)^{1/2}(I+\mathcal{P}_{21}\mathcal{P}_{21}^*)^{*/2} \\ I+\mathcal{P}_{12}^*\mathcal{P}_{12} &=& (I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{*/2}(I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2} \end{array} \right. . \tag{1.6.17}$$

Proof: Let us (primarily to simplify the notation in the cost function (1.6.13)) define $\mathcal{Q} = \mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}$. Note that we can now write the cost function as

$$\left\| \begin{bmatrix} \mathcal{P}_{11}+\mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{Q} \\ \mathcal{Q}\mathcal{P}_{21} & \mathcal{Q} \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} \mathcal{P}_{11}+\mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21} \\ \mathcal{Q}\mathcal{P}_{21} \end{bmatrix} \right\|_2^2 + \left\| \begin{bmatrix} \mathcal{P}_{12}\mathcal{Q} \\ \mathcal{Q} \end{bmatrix} \right\|_2^2. \tag{1.6.18}$$

Now using the solution to the full information control problem (1.6.16) it is straightforward to see that we may write (see *e.g.*, the proofs of Theorems 1.3.1 and 1.4.1)

$$\begin{aligned} \left\| \begin{bmatrix} \mathcal{P}_{11}+\mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21} \\ \mathcal{Q}\mathcal{P}_{21} \end{bmatrix} \right\|_2^2 &= \left\| (I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f - (I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{Q}\mathcal{P}_{21} \right\|_2^2 \\ &\quad + \text{terms independent of } \mathcal{Q}. \end{aligned}$$

Moreover, we have

$$\left\| \begin{bmatrix} \mathcal{P}_{12}\mathcal{Q} \\ \mathcal{Q} \end{bmatrix} \right\|_2^2 = \left\| (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{Q} \right\|_2^2 . \tag{1.6.19}$$

Combining these last two expressions shows that we need to minimize the cost function,

$$\left\| \begin{bmatrix} (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f - (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{Q}\mathcal{P}_{21} & -(I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{Q} \end{bmatrix} \right\|_2^2 . \tag{1.6.20}$$

But defining, $\mathcal{K}_e = (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{Q}$, yields the estimation problem (1.6.15). Solving $\mathcal{K}_e = (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}$ for the desired controller $\mathcal{K}$ gives our final result (1.6.14).

■

Although we have obtained the separation principle and the solution to the measurement feedback control problem through a purely algebraic route, the solution has certain physical significance that should be noted. In the first step we need to find the full information controller that satisfies

$$\min_{\text{causal } \mathcal{K}_f} \left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}_f \\ \mathcal{K}_f \end{bmatrix} \right\|_2 .$$

The above controller would have been the H^2-optimal controller if we had access to the exogenous input, w. (Recall that $s = \mathcal{P}_{11}w + \mathcal{P}_{12}u$.) Since we do not have access to w, we cannot construct the full information control signal, $u_f = \mathcal{K}_f w$. However, what the separation principle says is that we should estimate the signal u_f using the observations, $y = \mathcal{P}_{21}w + \mathcal{P}_{22}u + v$. (Note here that, since at each time instant the previous values of u are known, this is a standard estimation problem.) The exact statement of the resulting H^2 estimation problem is,

$$\min_{\text{causal } \mathcal{K}_e} \left\| \begin{bmatrix} (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f - \mathcal{K}_e\mathcal{P}_{21} & -\mathcal{K}_e \end{bmatrix} \right\|_2 ,$$

which means that we are estimation the signal $(I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f w$ using the observation, $y - \mathcal{P}_{22}u = \mathcal{P}_{21} + v$. [The last step (1.6.14) simply means that we should unwind our estimator, which was in terms of previous u and current and previous y, to make it an estimator in terms of y alone.]

As a matter of fact, we shall later see that in actually trying to find the H^2-optimal measurement feedback controller (say, for systems described by state-space models) it is easier to keep the physical meaning of the separation in mind, rather than to explicitly use the solution of Theorem 1.6.2.

Finally, we remark once more that the key to the solution of the H^2 full information and measurement feedback control problems is obtained through the canonical factorization of certain positive definite transfer operators.

1.6.4 Special Cases

We will now consider the solutions of Theorems 1.6.1 and 1.6.2 in the cases where the underlying transfer operators have transfer matrix representations and state-space structure. We shall, in particular, see that the in the full information problem the solution is given by a (well known) state feedback law, and that in the measurement feedback problem one instead uses a feedback based on estimates of the state.

Transfer Matrices - Full Information Case

As noted in Sec. 1.5.3, when the underlying transfer operators have transfer matrix representations we can write

$$\left\{ \begin{array}{rcl} s(z) & = & P_1(z)w(z) + P_2(z)u(z) \\ u(z) & = & K(z)w(z) \end{array} \right. . \tag{1.6.21}$$

Suppose, moreover, that the quadratic cost whose expected value we seek to minimize is given by

$$s_i^* R^c s_i + u_i^* Q^c u_i, \tag{1.6.22}$$

where $R_c \geq 0$ and $Q^c \geq 0$ are given weighting matrices.[19]

Using Theorem 1.6.1 the full information controller is given by

$$K(z) = -N^{-1}(z) R_c^{-1} \left\{ N^{-*}(z^{-*}) P_2^*(z^{-*}) R^c P_1(z) \right\}_+ , \tag{1.6.23}$$

[19] This slight generalization of the problem considered so far is of no significant consequence. Indeed we can go back to the standard problem by the transformations, $(R^c)^{*/2} s_i \to s_i$, $(Q^c)^{*/2} u_i \to u_i$, $(R^c)^{*/2} P_2(z) \to P_2(z)$ and $(R^c)^{*/2} P_1(z) \to P_1(z)$.

where $N(z)$ is found from the dual canonical factorization,

$$Q^c + P_2^*(z^{-*})R^c P_2(z) = N^*(z^{-*})R_c N(z), \tag{1.6.24}$$

with $N(z)$ causal and causally invertible, and where R_c is such that we have the normalization

$$N(\infty) = I_{m_1}. \tag{1.6.25}$$

As with the H^2 estimation problem, the solution requires a (here dual) canonical spectral factorization. As also was the case there, when the transfer matrices have state-space structure, viz.,

$$\begin{bmatrix} P_1(z) & P_2(z) \end{bmatrix} = L(zI - F)^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix}, \tag{1.6.26}$$

then the canonical factorization can be explicitly obtained via the solution of a DARE (discrete-time algebraic Riccati recursion). Indeed, if $\{F, G_2(Q^c)^{1/2}\}$ is stabilizable and $\{F, (R^c)^{*/2}L\}$ is detectable, then $N(z)$ in (1.6.24) is given by

$$N(z) = I_{m_2} + K_c(zI - F)^{-1}G_2 \quad , \quad K_c = R_c^{-1}G_2^* P^c F \quad , \quad R_c = Q^c + G_2^* P^c G_2 \tag{1.6.27}$$

where P^c is the unique positive semidefinite solution to the (dual) DARE,[20]

$$P^c = F^* P^c F + L^* R^c L - K_c^* R_c K_c. \tag{1.6.28}$$

Moreover, P^c is such that the matrix,

$$F_c \triangleq F - G_2 K_c, \tag{1.6.29}$$

is stable, which is in accordance with the fact that the inverse of $N(z)$,

$$N^{-1}(z) = I_{m_2} - K_c(zI - F + G_2K_c)^{-1}G_2 = I - K_c(zI - F_c)^{-1}G_2, \tag{1.6.30}$$

must be causal.

[20]Note that this DARE is the dual to the DARE (1.3.18) given in the solution to the H^2 estimation problem.

We are now in a position to give a more explicit formula for the H^2-optimal full information controller $K(z)$, of (1.6.23). To this end, note that

$$\begin{aligned} N^{-*}(z^{-*})P_2^*(z^{-*}) &= \left[I_{m_2} + G_2^*(z^{-1}I - F^*)^{-1}K_c^*\right]^{-1} G_2^*(z^{-1}I - F^*)^{-1}L^* \\ &= G_2^*(z^{-1}I - F^* + K_c^*G_2^*)^{-1}L^* \\ &= G_2^*(z^{-1}I - F_c^*)^{-1}L^*, \end{aligned}$$

so that we may write

$$\left\{N^{-*}(z^{-*})P_2^*(z^{-*})R^cP_1(z)\right\}_+ = \left\{G_2^*(z^{-1}I - F_c^*)^{-1}L^*R^cL(zI - F)^{-1}G_1\right\}_+ .$$

The causal part of the above expression can be found in a manner similar to what was done in the Wiener filtering setting. Using an argument analogous to the one used in proving Eq. (1.3.21), allows us to write

$$(z^{-1}I - F_c^*)^{-1}L^*R^cL(zI - F)^{-1} = P^c + P^cF(zI - F)^{-1} + (z^{-1}I - F_c^*)^{-1}F_c^*P^c, \quad (1.6.31)$$

which is the desired decomposition of $(z^{-1}I - F_c^*)^{-1}L^*R^cL(zI - F)^{-1}$ into its causal and anticausal parts. This then allows us to write

$$\left\{G_2^*(z^{-1}I - F_c^*)^{-1}L^*R^cL(zI - F)^{-1}G_1\right\}_+ = G_2^*P^cG_1 + G_2^*P^cF(zI - F)^{-1}G_1, \quad (1.6.32)$$

so that using (1.6.23) we finally obtain the desired expression for $K(z)$,

$$\begin{aligned} K(z) &= -N^{-1}(z)R_c^{-1}\left(G_2^*P^cG_1 + G_2^*P^cF(zI - F)^{-1}G_1\right)^{-1} \\ &= -R_c^{-1}G_2^*P^cG_1 - K_c(zI - F_c)^{-1}(I - G_2R_c^{-1}G_2^*P^c)G_1. \end{aligned} \quad (1.6.33)$$

where to obtain the second equality we have used the expression for $N^{-1}(z)$ from (1.6.30).

We can now use the above transfer matrix representation to write a state-space model for the controller as follows,

$$\left\{ \begin{aligned} x_{i+1} &= (F - G_2K_c)x_i + (I - G_2R_c^{-1}G_2^*P^c)G_1w_i \\ u_i &= -K_cx_i - R_c^{-1}G_2^*P^cG_1w_i \end{aligned} \right. . \quad (1.6.34)$$

The reason why we have used the notation x_i for the state variable in (1.6.34) is that it really is the state vector for the plant. Indeed using the expression for $u_i = -K_c x_i - R_c^{-1} G_2^* P^c G_1 w_i$ in the state equation of (1.6.34) allows us to write,

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F x_i + G_1 w_i + G_2 u_i \\ u_i & = & -K_c x_i - R_c^{-1} G_2^* P^c G_1 w_i \end{array} \right. , \tag{1.6.35}$$

which shows that x_i is the plant's state vector.

The above solution has an interesting structure and shows that the control signal, u_i, is a function *only* of the current state, x_i, and the current driving disturbance, w_i. This is slightly different from the (now famous) state feedback solution of continuous-time H^2 optimal control where the control signal depends only on the current state. The reason why this difference occurs is that (from the outset) we have insisted on causal controllers. Had we insisted on *strictly* causal controllers, *i.e.*, controllers that only have access to *past* values of the exogenous input, w, then we would have obtained a state feedback controller.

Indeed in that case, following similar arguments, one can show that the strictly causal H^2-optimal full information controller is given by,

$$K(z) = -N^{-1}(z) R_c^{-1} \left\{ N^{-*}(z^{-*}) P_2^*(z^{-*}) R^c P_1(z) \right\}_{s.c.} , \tag{1.6.36}$$

where the notation $\{A(z)\}_{s.c.}$ denotes the strictly causal part of the transfer matrix $A(z)$. In this case,

$$\left\{ G_2^*(z^{-1}I - F_c^*)^{-1} L^* R^c L (zI - F)^{-1} G_1 \right\}_{s.c.} = G_2^* P^c F (zI - F)^{-1} G_1,$$

from which we conclude

$$K(z) = K_c (zI - F_c)^{-1} G_1, \tag{1.6.37}$$

which has state-space representation,

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F x_i + G_1 w_i + G_2 u_i \\ u_i & = & -K_c x_i \end{array} \right. . \tag{1.6.38}$$

This last expression is the wellknown state feedback law of H^2-optimal control.

It is useful to summarize the results obtained so far in the following Theorem.

Theorem 1.6.3 (Infinite-Horizon H^2-optimal Full Information Controller) *(a) The solution to the problem*

$$\min_{\text{causal } K(z)} \left\| \begin{bmatrix} R^{*/2}\left(P_1(z) + P_2(z)K(z)\right) \\ Q^{*/2}K(z) \end{bmatrix} \right\|_2, \tag{1.6.39}$$

is given by

$$K(z) = -N^{-1}(z)R_c^{-1}\left\{N^{-*}(z^{-*})P_2^*(z^{-*})R^cP_1(z)\right\}_+, \tag{1.6.40}$$

where $N(z)$ is found from the dual canonical spectral factorization

$$Q^c + P_2^*(z^{-*})R^cP_2(z) = N^*(z^{-*})R_cN(z),$$

with $N(z)$ causal and causally invertible, and $N(\infty) = I_{m_1}$.

When $P_1(z)$ and $P_2(z)$ have state-space structure,

$$\begin{bmatrix} P_1(z) & P_2(z) \end{bmatrix} = L(zI - F)^{-1}\begin{bmatrix} G_1 & G_2 \end{bmatrix},$$

with $\{F, G_2(Q^c)^{1/2}\}$ stabilizable and $\{F, (R^c)^{*/2}L\}$ detectable, then

$$K(z) = -R_c^{-1}G_2^*P^cG_1 - K_c(zI - F_c)^{-1}(I - G_2R_c^{-1}G_2^*P^c)G_1, \tag{1.6.41}$$

where $K_c = R_c^{-1}G_2^*P^cF$, $R_c = Q^c + G_2^*P^cG_2$, and P^c the unique positive semidefinite solution of the DARE,

$$P^c = F^*PF + L^*R^cL - K_c^*R_cK_c.$$

In this case, the control signal, u_i, is given by

$$u_i = -K_cx_i - R_c^{-1}G_2^*P^cG_1w_i, \tag{1.6.42}$$

where x_i is the state variable satisfying, $x_{i+1} = Fx_i + G_1w_i + G_2u_i$.
(b) The solution to the problem

$$\min_{\text{strictly causal } K(z)} \left\| \begin{bmatrix} R^{*/2}\left(P_1(z) + P_2(z)K(z)\right) \\ Q^{*/2}K(z) \end{bmatrix} \right\|_2, \tag{1.6.43}$$

is given by

$$K(z) = N^{-1}(z)R_c^{-1}\left\{N^{-*}(z^{-*})P_2^*(z^{-*})R^cP_1(z)\right\}_{s.c.}. \tag{1.6.44}$$

When $P_1(z)$ and $P_2(z)$ have the state-space structure of part (a), then the H^2-optimal control signal is given by the state feedback law,

$$u_i = -K_c x_i, \tag{1.6.45}$$

where K_c is as in part (a).

Transfer Matrices - Measurement Feedback Case

In the measurement feedback case, the underlying transfer operators have transfer matrix representations of the form,

$$\left\{ \begin{array}{lcl} s(z) & = & P_{11}(z)w(z) + P_{12}(z)u(z) \\ y(z) & = & P_{21}(z)w(z) + P_{22}(z)u(z) + v(z) \\ u(z) & = & K(z)y(z) \end{array} \right. . \tag{1.6.46}$$

We shall once more assume that the cost function (whose expected value is to be minimized) is given by

$$s_i^* R^c s_i + u_i^* Q^c u, \tag{1.6.47}$$

and that the $\{w_i, v_i\}$ are zero-mean stationary stochastic processes such that

$$E \begin{bmatrix} w_i \\ v_i \end{bmatrix} \begin{bmatrix} w_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \delta_{ij}. \tag{1.6.48}$$

The H^2-optimal measurement feedback controller can now be found using the recipe of Theorem 1.6.2. However, for brevity we shall not do so here, since the solution is simply the restatement of Theorem 1.6.2 when transfer operators are replaced by transfer matrices. Instead we shall focus on the case where the $P_{ij}(z)$ have state-space structure of the form

$$\begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} = \begin{bmatrix} L \\ H \end{bmatrix} (zI - F)^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix}. \tag{1.6.49}$$

As we have just seen, the full information controller is given by

$$u_{f,i} = -K_c x_i - R_c^{-1} G_2^* P^c G_1 w_i.$$

Now according to Theorem 1.6.2 the H^2-optimal measurement feedback control signal is simply the H^2 estimate of $u_{f,i}$ given the observations, $\{y_j, j \leq i\}$. Thus,

$$u_i = -K_c \hat{x}_{i|i} - R_c^{-1} G_2^* P^c G_1 \hat{w}_{i|i}, \tag{1.6.50}$$

where $\hat{x}_{i|i}$ and $\hat{w}_{i|i}$ denote the linear least-mean-squares estimates of x_i and w_i given $\{y_j, j \leq i\}$.[21] But since the $\{y_j, j \leq i\}$ are independent of w_i (since $P_{21}(z) = H(zI - F)^{-1}G_1$ is strictly causal and the $\{w_j\}$ are white), we have $\hat{w}_{i|i} = 0$, so that

$$u_i = -K_c \hat{x}_{i|i}. \tag{1.6.51}$$

Since the observations, y_i, satisfy the state-space model,

$$\begin{cases} x_{i+1} &= F x_i + G_1 w_i + G_2 u_i \\ y_i &= H x_i + v_i \end{cases} \tag{1.6.52}$$

the state estimates can be readily found from the Wiener filter recursions (see Theorem 1.3.2),

$$\begin{cases} \hat{x}_{i+1} &= (F - K_p H)\hat{x}_i + K_p y_i + G_2 u_i \\ \hat{x}_{i|i} &= (I - PH^* R_e^{-1} H)\hat{x}_i + PH^* R_e^{-1} y_i \end{cases}, \tag{1.6.53}$$

where $K_p = FPH^*R_e^{-1}$, $R_e = R + HPH^*$, P is the unique positive semidefinite solution to the DARE,

$$P = FPF^* + G_1 Q G_1^* - K_p R_e K_p^*, \tag{1.6.54}$$

and where we have used the fact that at time i, the $\{u_j, j < i\}$ are known. Using, $u_i = -K_c \hat{x}_{i|i}$ The above state-space model may be rewritten as

$$\begin{cases} \hat{x}_{i+1} &= \left(F - K_p H - G_2 K_c (I - PH^* R_e^{-1} H)\right) \hat{x}_i + (K_p - G_2 K_c PH^* R_e^{-1}) y_i \\ u_i &= -K_c (I - PH^* R_e^{-1} H)\hat{x}_i - K_c PH^* R_e^{-1} y_i \end{cases}, \tag{1.6.55}$$

from which we infer that optimal controller is

$$K(z) = -K_c \left[PH^* R_e^{-1} + (I - PH^* R_e^{-1} H)(zI - F_m)^{-1} (K_p - G_2 K_c PH^* R_e^{-1}) \right], \tag{1.6.56}$$

[21]Note that we have used the fact that in H^2 estimation the best estimate of any linear combination of the state is simply that linear combination of the best state estimate. We saw that this was not the case in H^∞ estimation: a fact that will have certain ramifications for H^∞ control.

where we have defined, $F_m \triangleq F - K_pH - G_2K_c(I - PH^*R_e^{-1}H)$.

A simpler expression for the controller results if we insist on a strictly causal $K(z)$, in which case,

$$u_i = -K_c\hat{x}_i, \tag{1.6.57}$$

where $\hat{x}_i$ is the H^2-optimal estimate of x_i given $\{y_j, j < i\}$ and satisfies the predicted form of the Wiener filter,

$$\hat{x}_{i+1} = (F - K_pH)\hat{x}_i + K_py_i + G_2u_i = (F - K_pH - G_2K_c)\hat{x}_i + K_py_i. \tag{1.6.58}$$

In this case, $K(z)$, is given by

$$K(z) = -K_c(zI - F + K_pH + G_2K_c)^{-1}K_p. \tag{1.6.59}$$

We summarize the results obtained in the following theorem.

Theorem 1.6.4 (Infinite-Horizon H^2-optimal Measurement Feedback Controller) *Consider the state-space model*

$$\begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} = \begin{bmatrix} L \\ H \end{bmatrix} (zI - F)^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix},$$

with $\{F, G_1Q^{1/2}\}$ and $\{F, G_2(Q^c)^{1/2}\}$ stabilizable and $\{F, H\}$ and $\{F, (R^c)^{*/2}L\}$ detectable, and consider the transfer matrix,

$$T_K = \begin{bmatrix} (R^c)^{*/2}\left(P_{11} + P_{12}K\left(I - P_{22}K\right)^{-1}P_{21}\right)Q^{1/2} & (R^c)^{*/2}\left(P_{12}K\left(I - P_{22}K\right)^{-1}\right)R^{1/2} \\ (Q^c)^{*/2}\left(K\left(I - P_{22}K\right)^{-1}P_{21}\right)Q^{1/2} & (Q^c)^{*/2}\left(K\left(I - P_{22}K\right)^{-1}\right)R^{1/2} \end{bmatrix},$$

where for notational simplicity we have suppressed the dependence of the transfer matrices on z.

(a) The solution to the problem

$$\min_{\text{causal } K(z)} \|T_K(z)\|_2$$

is given by

$$K(z) = -K_c\left[PH^*R_e^{-1} + (I - PH^*R_e^{-1}H)(zI - F_m)^{-1}(K_p - G_2K_cPH^*R_e^{-1})\right],$$

where $F_m = F - K_pH - G_2K_c(I - PH^*R_e^{-1}H)$, $K_p = FPH^*R_e^{-1}$, $R_e = R + HPH^*$, $K_c = R_c^{-1}G_2^*P^cF$, $R_c = Q^c + G_2^*P^cG_2$, *and where* P *and* P^c *are the unique positive semidefinite solutions to the DARE's,*

$$\left\{ \begin{array}{rcl} P & = & FPF^* + G_1QG_1^* - K_pR_eK_p^* \\ P^c & = & F^*P^cF + L^*R^cL - K_c^*R_cK_c \end{array} \right. .$$

In particular, the control signal is given by

$$u_i = -K_c\left[(I - PH^*R_e^{-1}H)\hat{x}_i + PH^*R_e^{-1}y_i\right],$$

where

$$\hat{x}_{i+1} = (F - K_pH)\hat{x}_i + K_py_i + G_2u_i.$$

(b) The solution to the problem

$$\min_{\text{strictly causal } K(z)} \|T_K(z)\|_2$$

is given by

$$K(z) = -K_c(zI - F + K_pH + G_2K_c)^{-1}K_p,$$

where K_p *and* K_c *are as in part (a). In particular, the control signal is given by*

$$u_i = -K_c\hat{x}_i,$$

where

$$\hat{x}_{i+1} = (F - K_pH)\hat{x}_i + K_py_i + G_2u_i.$$

State-Space Models

Consider now the finite horizon case and suppose that the transfer operators have (possibly time-varying) state-space models which for the full information problem we may write as

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F_ix_i + G_{1,i}w_i + G_{2,1}u_i \\ s_i & = & L_ix_i \end{array} \right. , x_0, \quad 0 \leq i \leq N \tag{1.6.60}$$

and for the measurement feedback problem,

$$\begin{cases} x_{i+1} &= F_i x_i + G_{1,i} w_i + G_{2,1} u_i \\ s_i &= L_i x_i \\ y_i &= H_i x_i + v_i \end{cases}, x_0, \quad 0 \leq i \leq N. \tag{1.6.61}$$

Moreover, it is assumed that x_0 and the $\{u_i\}$ and $\{v_i\}$ are zero-mean uncorrelated random variables with known covariance matrices

$$E \begin{bmatrix} x_0 \\ u_i \\ v_i \end{bmatrix} \begin{bmatrix} x_0^* & u_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i \delta_{ij} & 0 \\ 0 & 0 & R_i \delta_{ij} \end{bmatrix}, \tag{1.6.62}$$

and that the cost function whose expected value is to be minimized is given by

$$\sum_{i=0}^{N} s_i^* R_i^c s_i + \sum_{i=0}^{N} u_i^* Q_i^c u_i + x_{N+1}^* P_{N+1}^c x_{N+1}, \tag{1.6.63}$$

where $R_i^c \geq 0$, $Q_i^c \geq 0$ and $P_{N+1}^c \geq 0$ are given weighting matrices.

We shall not give the finite horizon full information and measurement feedback controllers here since they are very similar to the solutions given by Theorems 1.6.3 and 1.6.4. Essentially, the only difference is that the DARE's for P and P^c are replaced by Riccati recursions for the time-varying matrices, P_i and P_i^c, *i.e.*,

$$P_{i+1} = F_i P_i F_i^* + G_{1,i} Q_i G_{1.i}^* - K_{p,i} R_{e,i} K_{p,i}^*, \quad P_0 = \Pi_0 \tag{1.6.64}$$

where $K_{p,i} = F_i P_i H_i^* R_{e,i}^{-1}$ and $R_{e,i} = R_i + H_i P_i H_i^*$, and

$$P_i^c = F_i^* P_{i+1}^c F_i + L_i^* R_i^c L_i - K_{c,i}^* R_{c,i} K_{c,i}, \quad P_{N+1}^c \tag{1.6.65}$$

where $K_{c,i} = R_{c,i}^{-1} G_{2,i}^* P_{i+1}^c F$ and $R_{c,i} = Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i}$.

We end this section with two comments. First, as we shall in Chapter 2, the above Riccati recursions recursively perform the canonical and dual canonical factorizations of the positive definite Gramians, $I + \mathcal{P}_{21}\mathcal{P}_{21}^*$ and $I + \mathcal{P}_{12}^*\mathcal{P}_{12}$. Second, when the system matrices are constant and as time progresses to infinity, it is an issue whether or not the Riccati recursion solutions converge to the unique positive semidefinite solutions of their corresponding DARE's (in other words, whether or not the finite horizon controllers converge to their infinite horizon counterparts). This issue will be taken up in Chapter 8.

1.6.5 The Question of Robustness

As we have just seen, under suitable stochastic assumptions, such as (jointly Gaussian) uncorrelated disturbances with known first and second order statistics, H^2-optimal controllers have certain desirable optimality properties, namely that they minimize the expected value of a certain quadratic cost.

In practice, however, we may not always know the statistics of the disturbances. We also often do not know the exact models that generate our signals of interest. Therefore, as was the case with H^2 estimation, we cannot always guarantee the validity of the assumptions required of H^2 controllers. The natural question that arises in this regard is what the performance of such controllers will be if the assumptions on the disturbances are violated, or if there are modeling errors in our model so that the disturbances must include the modeling errors? In other words,

- *is it possible that* **small** *disturbances and modeling errors may lead to* **large** *costs?*

Intuitively, a non-robust controller is one for which the above is true, *i.e.*, one for which small disturbances may force us to incur large costs, whereas a robust controller is one for which small disturbances always result in small costs.

In the H^∞ framework to robust control the costs (both the objective cost and the cost associated with the disturbances) are quadratic[22] forms constructed from the signals, and the goal is to come up with controllers that minimize (or in the suboptimal case, bound) the maximum gain from the disturbance cost to the objective cost. This will guarantee that if the disturbances are small, *i.e.*, if their associated cost is small, then the objective cost will be as small as possible, *no matter what the disturbances are*. The robustness of H^∞ controllers, with respect to modeling errors and disturbance variation, follows from the fact that the maximum gain is minimized over *all possible* disturbances. However, since they make no assumption about the disturbances and have to accommodate for all conceivable ones, they may be over-conservative.

Although the aforementioned motivation for H^∞ control is an important one,

[22]Other approaches to robust control differ in the definition of the cost. In l_1 control the cost used is related to the peak value of the signals, [DP87] and [DDB95].

historically, the main motivation for its introduction was somewhat different and is related to what is now called the robust stabilization problem. [See the pioneering paper [Zam81] for the initial motivation of H^∞ control and the textbooks [Fra87], [GL95] and [ZDG96] for further properties and related topics.] To understand the robust stabilization problem[23] consider the feedback structure of Fig. 1.5.

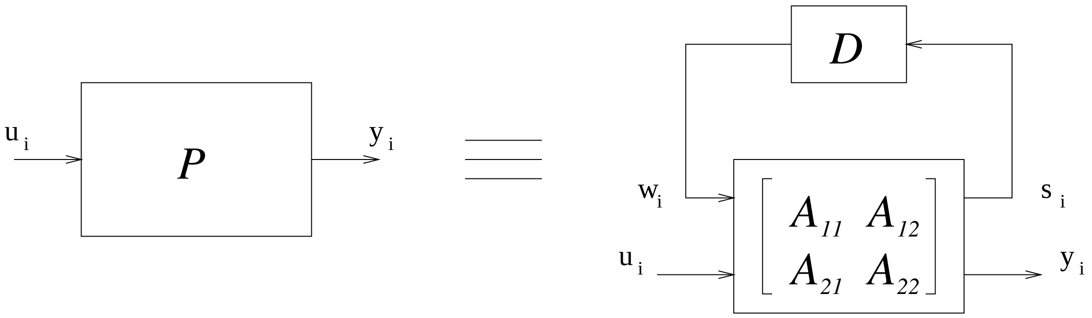

Figure 1.5: A plant with modeling error.

Suppose that we have a plant, $\mathcal{P}$, that can be represented by a known nominal plant, $\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$, along with an *unknown* plant, D, that are connected in a feedback form. In this setting $\mathcal{A}$ represents that part of the plant model that we are aware of, and $\mathcal{D}$ represents the modeling errors. Although the uncertainty model, $\mathcal{D}$, is unknown we shall assume that it is 'small' (in a sense to be defined shortly).

As depicted in Fig. 1.6, the objective here is to design a controller, $\mathcal{K}$, that stabilizes $\mathcal{P}$ (for all small uncertainty $\mathcal{D}$). Since we know the nominal plant, we can always design a controller that stabilizes it (indeed our ultimate design must also stabilize $\mathcal{A}$ since it corresponds to the special case, $\mathcal{D} = 0$), say, an H^2 controller. However, it is not in general clear how robust such a control strategy would be to the uncertainty, $\mathcal{D}$. In other words, is it possible that 'small' modeling errors (which in practice are inevitable) may unstabilize the composite plant?

To answer this question, let us persist with some choice of controller, $\mathcal{K}$, and represent the composite plant as the feedback connection of the uncertainty, $\mathcal{D}$, and

[23]See also [San78] for what seems to be the first analysis of this problem.

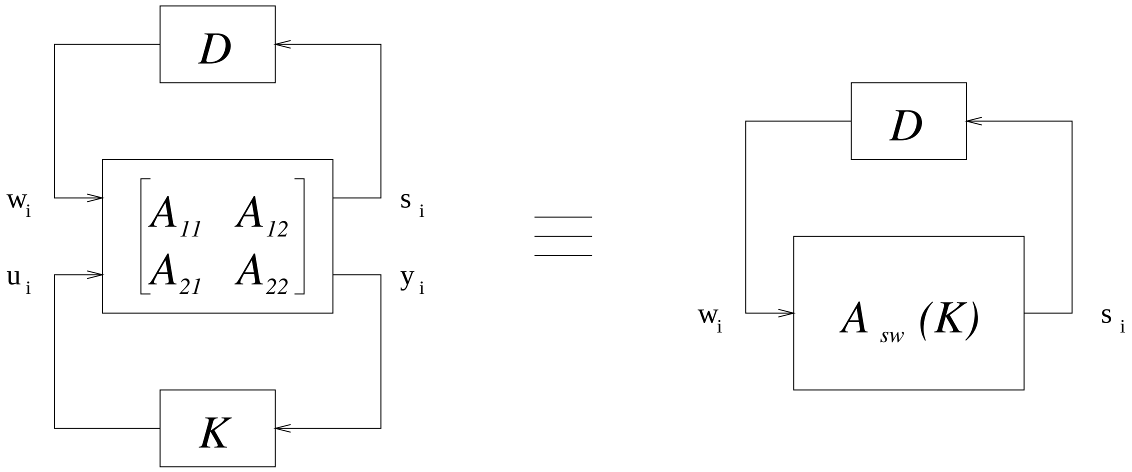

Figure 1.6: The robust stabilization problem.

the known plant,

$$\mathcal{A}_{sw}(\mathcal{K}) = \mathcal{A}_{11} + \mathcal{A}_{12}\mathcal{K}(I - \mathcal{A}_{22}\mathcal{K})^{-1}\mathcal{A}_{21}. \tag{1.6.66}$$

Now the composite plant will be input-output stable in some given norm, $\|\cdot\|$, if, according to the *small gain theorem* (see *e.g.*, [Wil71a], [DV75]), we have

$$\|\mathcal{D}\mathcal{A}_{sw}(\mathcal{K})\| < 1. \tag{1.6.67}$$

But if we choose a submultiplicative norm, such as the H^∞ norm,[24] $\|\cdot\|_\infty$, the composite plant will be stable if,

$$\|\mathcal{D}\|_\infty \cdot \|\mathcal{A}_{sw}(\mathcal{K})\|_\infty < 1. \tag{1.6.68}$$

Suppose now that we know that $\mathcal{D}$ is small in the sense that

$$\|\mathcal{D}\|_\infty < \epsilon. \tag{1.6.69}$$

Then we will have stability if,

$$\|\mathcal{A}_{sw}(\mathcal{K})\|_\infty < \frac{1}{\epsilon} \triangleq \gamma. \tag{1.6.70}$$

Indeed, we even have the following stronger statement. If (1.6.70) is not satisfied, then there exists some norm-bounded uncertainty, $\mathcal{D}$ with $\|\mathcal{D}\|_\infty < \epsilon$, such that

[24] We could also use other submultiplicative norms such as the l_1 norm.

the composite plant is unstable. Therefore condition, (1.6.70) is the necessary and sufficient condition for input-output stability of the composite plant for all model uncertainties, $\|\mathcal{D}\|_\infty < \epsilon$.

Thus we are lead to the design of a controller that bounds the H^∞ norm from the exogenous input w to the regulated output s for the nominal plant. We will study the H^∞ control problem in a more general setting (one which subsumes the above problem) that is motivated by bounding the maximum gain from the disturbance cost to the objective cost.

1.7 The H^∞ Approach

In this section we briefly describe the H^∞ approach to robust control. [For alternative presentations and derivations see the textbooks [Fra87], [BB95], [GL95], [ZDG96] and the references therein.]

Returning to the control problems of Sec. 1.5, we recall that a useful representation for any control strategy $\mathcal{K}$ is, in the full information problem, the transfer operator,

$$\mathcal{T}^c_{\mathcal{K}} = \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix},$$

that maps the exogenous input $\{w_j\}$ to the regulated and control signals $\{s_j\}$ and $\{u_j\}$, and, in the measurement feedback problem, the transfer operator,

$$\mathcal{T}^c_{\mathcal{K}} = \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix},$$

that maps the exogenous inputs $\{w_j\}$ and $\{v_j\}$ to the regulated and control signals $\{s_j\}$ and $\{u_j\}$.

In either case, for any choice of controller $\mathcal{K}$, and for any exogenous input sequence, we can readily compute the energy gain from the exogenous inputs to the regulated and control signals. For example in the measurement feedback problem,

$$\frac{\|s\|_2^2 + \|u\|_2^2}{\|w\|_2^2 + \|v\|_2^2} = \frac{\left\| \mathcal{T}^c_{\mathcal{K}} \begin{bmatrix} w \\ v \end{bmatrix} \right\|_2^2}{\|w\|_2^2 + \|v\|_2^2}. \tag{1.7.1}$$

Clearly, the ratio in (1.4.1) depends on the particular choice of the input disturbances w and v. As in the H^∞ estimation problem, to remove this dependency, and to ensure robustness with respect to disturbance variation, we consider the largest energy gain in (1.7.1) over all possible w and v, *i.e.*, the H^∞ norm of a transfer operator $\mathcal{T}_\mathcal{K}^c$. This leads us to the following problem.

Problem 1.7.1 (Optimal H^∞ Control Problems) *(a)* Full Information Problem: *Find a causal controller $\mathcal{K}$ that minimizes the H^∞ norm of the transfer operator $\mathcal{T}_\mathcal{K}^c$ that maps the exogenous input $\{w_j\}$ to the regulated and control signals, $\{s_j\}$ and $\{u_j\}$,* i.e. , *find a causal $\mathcal{K}$ that satisfies,*

$$\inf_{\mathcal{K}} \|\mathcal{T}_\mathcal{K}^c\|_\infty = \inf_{\mathcal{K}} \left\| \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix} \right\|_\infty . \tag{1.7.2}$$

Moreover find the resulting $\gamma_{opt} = \inf_\mathcal{K} \|\mathcal{T}_\mathcal{K}^c\|_\infty$.

(b) Measurement Feedback Problem: *Find a causal controller $\mathcal{K}$ that minimizes the H^∞ norm of the transfer operator $\mathcal{T}_\mathcal{K}^c$ that maps the exogenous inputs $\{w_j\}$ and $\{v_j\}$ to the regulated and control signals, $\{s_j\}$ and $\{u_j\}$,* i.e. , *find a causal $\mathcal{K}$ that satisfies,*

$$\inf_{\mathcal{K}} \|\mathcal{T}_\mathcal{K}^c\|_\infty = \inf_{\mathcal{K}} \left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix} \right\|_\infty . \tag{1.7.3}$$

Moreover find the resulting $\gamma_{opt} = \inf_\mathcal{K} \|\mathcal{T}_\mathcal{K}^c\|_\infty$.

[Note, once more, that in the above problem statement we are purposefully ambiguous as to whether we are considering the finite horizon or (semi)infinite horizon case.]

The minimax nature of H^∞ optimal controllers is evident from (1.7.2) and (1.7.3). Thus the H^∞ control problem can be regarded as a game where nature (the opponent) has access to the unknown exogenous inputs w and v and we have choice of the causal controller $\mathcal{K}$. Note, moreover, that although H^∞ optimal controllers are robust with respect to disturbance variation they may be over conservative (as no assumption is made regarding the disturbances).

There are very few case where a closed-form solution to the optimal H^∞ problem of Prob. 1.7.1 can be found, and in general one resorts to the following suboptimal solution.

Problem 1.7.2 (Suboptimal H^∞ Control Problems) *(a)* Full Information Problem: *Given a $\gamma > 0$, find a causal estimator $\mathcal{K}$ that guarantees*

$$\|\mathcal{T}^c_{\mathcal{K}}\|_\infty = \left\| \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix} \right\|_\infty < \gamma. \tag{1.7.4}$$

This clearly requires checking whether γ is an achievable bound.

(b) Measurement Feedback Problem: *Given a $\gamma > 0$, find a causal estimator $\mathcal{K}$ that guarantees*

$$\|\mathcal{T}^c_{\mathcal{K}}\|_\infty = \left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix} \right\|_\infty < \gamma. \tag{1.7.5}$$

This clearly requires checking whether γ is an achievable bound.

1.7.1 The Full Information Solution

The solution to the full information H^∞ control problem is essentially the dual to the H^∞ estimation problem of Theorem 1.4.1 and is given below.

Theorem 1.7.1 (H^∞ Suboptimal Full Information Controllers) *A causal controller, $\mathcal{K}$ that achieves*

$$\left\| \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix} \right\|_\infty < \gamma$$

exists if, and only if, there exists a dual canonical factorization of the form

$$\begin{bmatrix} I + \mathcal{P}_2^*\mathcal{P}_2 & \mathcal{P}_2^*\mathcal{P}_1 \\ \mathcal{P}_1^*\mathcal{P}_2 & -\gamma^2 I + \mathcal{P}_1^*\mathcal{P}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix},$$

with $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ and $\mathcal{L}_{11}$ causal and causally invertible, and $\mathcal{L}_{21}$ strictly causal. If this is the case, then all possible H^∞ estimators of level γ are given by

$$\mathcal{K} = (\mathcal{L}_{11} - \mathcal{Q}\mathcal{L}_{21})^{-1}(\mathcal{Q}\mathcal{L}_{22} - \mathcal{L}_{12}), \tag{1.7.6}$$

where $\mathcal{Q}$ is any causal and strictly contractive operator. An important choice results from taking $\mathcal{Q} = 0$, so that

$$\mathcal{K}_{cen} = -\mathcal{L}_{11}^{-1}\mathcal{L}_{12}, \tag{1.7.7}$$

which is the so-called "central" controller.

Proof: The proof is the dual to the proof of Theorem 1.4.1 and will not be repeated here.

∎

Comparing the results of Theorems 1.4.1 and 1.7.1 reveals the duality between H^∞ estimation and H^∞ full information control. We shall not comment on this duality here (other than, perhaps, noting the replacement of an indefinite canonical factorization with a dual one), and shall instead close this section with a remark on the maximum entropy property of the central full information controller.

Corollary 1.7.1 (Maximum Entropy Property of Central Solution) *The solution to the problem*

$$\max_{\text{causal } \mathcal{K}} \log \det \left[\gamma^2 I - \mathcal{T}_{\mathcal{K}}^{c*}\mathcal{T}_{\mathcal{K}}^{c}\right] \tag{1.7.8}$$

is given by the central controller,

$$\mathcal{K}_{cen} = -\mathcal{L}_{11}^{-1}\mathcal{L}_{12}. \tag{1.7.9}$$

1.7.2 The Measurement Feedback Solution

The solution to the H^∞ measurement feedback control problem is given below and, as mentioned earlier, involves a certain separation principle. The separation principle for H^∞ control was first derived in [DGKF89] and is essentially the same as the separation principle of risk-sensitive control (see [Whi90]), although the form given below somewhat differs from those of [DGKF89] and [Whi90]. We should also remark that the H^∞ separation principle is different from its H^2 counterpart since the H^∞ full information control problem, and the H^∞ estimation problem, that the H^∞ measurement feedback problem breaks into are no longer decoupled. We will have more to say about this in a moment. We also note that, as in the H^2 case, we can no longer guarantee the stability of the controller. Moreover, for notational simplicity we have taken $\gamma = 1$.

Theorem 1.7.2 (H^∞ Suboptimal Measurement Feedback Controllers) *A causal controller $\mathcal{K}$ that achieves*

$$\left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix} \right\|_\infty < 1 \tag{1.7.10}$$

exists if, and only if, the following two conditions are satisfied.

(i) The full information control problem,

$$\left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}_f \\ \mathcal{K}_f \end{bmatrix} \right\|_\infty < 1 \tag{1.7.11}$$

has a causal solution, $\mathcal{K}_f$. In other words, there exists the dual canonical factorization,

$$\begin{bmatrix} I + \mathcal{P}_{12}^*\mathcal{P}_{12} & \mathcal{P}_{12}^*\mathcal{P}_{11} \\ \mathcal{P}_{11}^*\mathcal{P}_{12} & -I + \mathcal{P}_{11}^*\mathcal{P}_{11} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}, \tag{1.7.12}$$

with $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ and $\mathcal{L}_{11}$ causal and causally invertible, and $\mathcal{L}_{21}$ strictly causal.

(ii) The estimation problem,

$$\left\| \begin{bmatrix} \mathcal{L}_{12}\mathcal{L}_{22}^{-1} - \mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1} & -\mathcal{K}_e \end{bmatrix} \right\|_\infty < 1 \tag{1.7.13}$$

has a causal solution, $\mathcal{K}_e$, where the $\mathcal{L}_{ij}$ are as in part (i). In other words, there exists the canonical factorization,

$$\begin{bmatrix} I + \mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{22}^{-*}\mathcal{P}_{21}^* & \mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{22}^{-*}\mathcal{L}_{12}^* \\ \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{22}^{-*}\mathcal{P}_{21}^* & -I + \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{22}^{-*}\mathcal{L}_{12}^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11}^e & \mathcal{L}_{12}^e \\ \mathcal{L}_{21}^e & \mathcal{L}_{22}^e \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}^{e*} & \mathcal{L}_{21}^{e*} \\ \mathcal{L}_{12}^{e*} & \mathcal{L}_{22}^{e*} \end{bmatrix} \tag{1.7.14}$$

with $\begin{bmatrix} \mathcal{L}_{11}^e & \mathcal{L}_{12}^e \\ \mathcal{L}_{21}^e & \mathcal{L}_{22}^e \end{bmatrix}$ and $\mathcal{L}_{11}^e$ causal and causally invertible, and $\mathcal{L}_{12}^e$ strictly causal.

If this is the case, then all possible measurement feedback H^∞ controllers are given by,

$$\mathcal{K} = (I + \mathcal{Q}\mathcal{P}_{22})^{-1}\mathcal{Q}, \tag{1.7.15}$$

where

$$\mathcal{Q} = -\mathcal{K}_e\left[\mathcal{L}_{11} - (\mathcal{L}_{12} - \mathcal{K}_e\mathcal{P}_{21})\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\right]^{-1}, \tag{1.7.16}$$

and $\mathcal{K}_e$ is the solution to the estimation problem (1.7.13) which is parametrized by

$$\mathcal{K}_e = (\mathcal{L}_{22}^e\mathcal{S} - \mathcal{L}_{21}^e)(\mathcal{L}_{11}^e - \mathcal{L}_{12}^e\mathcal{S})^{-1}, \tag{1.7.17}$$

with $\mathcal{S}$ any causal and strictly contractive operator. The so-called central controller, $\mathcal{K}_{cen}$ results from the choice $\mathcal{S} = 0$ and corresponds to $\mathcal{K}_{e,cen} = -\mathcal{L}_{21}^e(\mathcal{L}_{11}^e)^{-1}$.

Proof: Recall the defining relations

$$\begin{cases} s &= \mathcal{P}_{11}w + \mathcal{P}_{12}u \\ y &= \mathcal{P}_{21}w + \mathcal{P}_{22}u + v \\ u &= \mathcal{K}y \end{cases}$$

so that we may write, $u = \mathcal{Q}\mathcal{P}_{21}u + \mathcal{Q}v$, with $\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}$. Now the condition (1.7.10) is satisfied if, and only if, for all u and v in h^2,

$$\frac{\|s\|_2^2 + \|u\|_2^2}{\|w\|_2^2 + \|v\|_2^2} < 1 \iff \|s\|_2^2 + \|u\|_2^2 - \|w\|_2^2 - \|v\|_2^2 < 0. \tag{1.7.18}$$

Defining, $J \triangleq \|s\|_2^2 + \|u\|_2^2 - \|w\|_2^2 - \|v\|_2^2$, we may write,

$$\begin{aligned} J &= \|\mathcal{P}_{11}w + \mathcal{P}_{12}u\|_2^2 + \|u\|_2^2 - \|w\|_2^2 - \|v\|_2^2 \\ &= \begin{bmatrix} u^* & w^* \end{bmatrix} \begin{bmatrix} I + \mathcal{P}_{12}^*\mathcal{P}_{12} & \mathcal{P}_{12}^*\mathcal{P}_{11} \\ \mathcal{P}_{11}^*\mathcal{P}_{12} & -\gamma^2 I + \mathcal{P}_{11}^*\mathcal{P}_{11} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} - \|v\|_2^2 \\ &= \begin{bmatrix} u^* & w^* \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} - \|v\|_2^2 \\ &= \|\mathcal{L}_{11}u + \mathcal{L}_{12}w\|_2^2 - \|\mathcal{L}_{21}u + \mathcal{L}_{22}w\|_2^2 - \|v\|_2^2 < 0, \end{aligned}$$

where in the third step we have used the obvious fact that if a measurement feedback controller exists, if, and only if, a full information one does, in which case dual canonical factorization can be performed. But this last inequality implies,

$$\frac{\|\mathcal{L}_{11}u + \mathcal{L}_{12}w\|_2^2}{\|\mathcal{L}_{21}u + \mathcal{L}_{22}w\|_2^2 + \|v\|_2^2} < 1.$$

We can now conclude that a measurement feedback controller exists if, and only if, $\mathcal{K}$ can be chosen such that,

$$\left\| \mathcal{T}_{\mathcal{K}}^{ba} \right\| < 1, \tag{1.7.19}$$

where $\mathcal{T}_{\mathcal{K}}^{ba}$ is the transfer operator that maps the signals b and v to the signal a, *i.e.*,

$$\mathcal{T}_{\mathcal{K}}^{ba} : \begin{bmatrix} b \\ v \end{bmatrix} \to a, \tag{1.7.20}$$

where we have defined,

$$\begin{cases} a &= \mathcal{L}_{11}u + \mathcal{L}_{12}w \\ b &= \mathcal{L}_{21}u + \mathcal{L}_{22}w \end{cases} \tag{1.7.21}$$

Let us now identify $\mathcal{T}_{\mathcal{K}}^{ba}$. To this end, note that

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{Q}\mathcal{P}_{21} & \mathcal{Q} \\ I & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{12} & \mathcal{L}_{11}\mathcal{Q} \\ \mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{22} & \mathcal{L}_{21}\mathcal{Q} \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}. \end{aligned}$$

Solving for w from the second of the above set of equations yields,

$$w = (\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{22})^{-1}b - (\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{22})^{-1}\mathcal{L}_{21}\mathcal{Q}v.$$

Plugging the above expression into the equation $a = (\mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{12})w + \mathcal{L}_{11}\mathcal{Q}v$ allows us to obtain

$$\mathcal{T}_{\mathcal{K}}^{ba} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_1 \end{bmatrix}, \tag{1.7.22}$$

where the entries of $\mathcal{T}_{\mathcal{K}}^{ba}$ are given by

$$\begin{cases} \mathcal{A}_1 &= (\mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{12})(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{22})^{-1} \\ \mathcal{A}_2 &= -(\mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{12})(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{22})^{-1}\mathcal{L}_{21}\mathcal{Q} + \mathcal{L}_{11}\mathcal{Q} \end{cases}. \tag{1.7.23}$$

As given above, the dependence of $\mathcal{T}_{\mathcal{K}}^{ba}$ on $\mathcal{Q}$ seems quite complicated and it does not appear clear how to choose $\mathcal{Q}$ (and hence $\mathcal{K}$) to make $\mathcal{T}_{\mathcal{K}}^{ba}$ a strict contraction.

However, let us persist and attempt to simplify the entries of $\mathcal{T}_{\mathcal{K}}^{ba}$. Indeed,

$$
\begin{aligned}
\mathcal{A}_2 &= \mathcal{L}_{11}\mathcal{Q}\left[-\mathcal{P}_{21}(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}+\mathcal{L}_{22})^{-1}\mathcal{L}_{21}\mathcal{Q}+I\right]-\mathcal{L}_{12}(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}+\mathcal{L}_{22})^{-1}\mathcal{L}_{21}\mathcal{Q} \\
&= \mathcal{L}_{11}\mathcal{Q}\left[I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\right]^{-1}-\mathcal{L}_{12}\mathcal{L}_{22}^{-1}(I+\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}\mathcal{L}_{22}^{-1})\mathcal{L}_{21}\mathcal{Q} \\
&= \mathcal{L}_{11}\mathcal{Q}(I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1}-\mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}(I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1} \\
&= (\mathcal{L}_{11}-\mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21})\mathcal{Q}(I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1} \\
&\triangleq -\mathcal{K}_e.
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
\mathcal{A}_1 &= \mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21}(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}+\mathcal{L}_{22})^{-1}+\mathcal{L}_{12}(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}+\mathcal{L}_{22})^{-1} \\
&= \mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1}+\mathcal{L}_{12}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} \\
&= \mathcal{L}_{11}\mathcal{Q}(I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1}\mathcal{P}_{21}\mathcal{L}_{22}^{-1}+\mathcal{L}_{12}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} \\
&= -\mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1}+\mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}(I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1}\mathcal{P}_{21}\mathcal{L}_{22}^{-1}+\mathcal{L}_{12}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} \\
&= -\mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1}+\mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1}+\mathcal{L}_{12}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} \\
&= -\mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1}+\mathcal{L}_{12}\left[\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}+I\right](I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} \\
&= -\mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1}+\mathcal{L}_{12}\mathcal{L}_{22}^{-1}.
\end{aligned}
$$

Thus we have shown,

$$
\mathcal{T}_{\mathcal{K}}^{ba}=\left[\begin{array}{cc}\mathcal{L}_{12}\mathcal{L}_{22}^{-1}-\mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1} & -\mathcal{K}_e\end{array}\right]. \tag{1.7.24}
$$

In other words, the measurement feedback problem has a solution if, and only if, $\mathcal{K}_e$ can be chosen such that $\mathcal{T}_{\mathcal{K}}^{ba}$ is a strict contraction. But this is simply the estimation problem (1.7.13). Finally, we note that solving $\mathcal{K}_e=(\mathcal{L}_{11}-\mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21})\mathcal{Q}(I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1}$ for $\mathcal{Q}$, and solving $\mathcal{Q}=\mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}$ for $\mathcal{K}$, yields the desired results (1.7.15) and (1.7.16).

■

A few remarks on the above result are in order. Note that the separation principle obtained states that to solve the problem we first need to solve the full information control problem,

$$
\left\|\left[\begin{array}{c}\mathcal{P}_{11}+\mathcal{P}_{12}\mathcal{K}_f \\ \mathcal{K}_f\end{array}\right]\right\|_\infty<\gamma, \tag{1.7.25}
$$

and then the estimation problem,

$$\left\| \begin{bmatrix} \mathcal{L}_{12}\mathcal{L}_{22}^{-1} - \mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1} & -\mathcal{K}_e \end{bmatrix} \right\|_\infty < \gamma. \tag{1.7.26}$$

Note, however, that, unlike the H^2 case, the above two problems are coupled, since the estimation problem involves quantities that have to be found from the full information problem. As a matter of fact, not only does the solution to the estimation problem depend on the full information controller, but the actual condition for the existence of the estimator depends on it as well. [This dependence occurs through the transfer operators $\mathcal{L}_{12}$ and $\mathcal{L}_{22}$ which are in turn determined by $\mathcal{P}_{11}$ and $\mathcal{P}_{12}$.] This is unlike the H^2 problem where the separation principle completely decouples the full information and estimation problems. [There it was neither necessary to decide on the noise covariances before designing the full information controller, nor on the control objectives before designing the optimal estimator.]

As we shall shortly see, in the state-space case, the coupling between the full information control and estimation problems is exemplified by the fact that their corresponding DARE's (or Riccati recursions in the finite-horizon case) are coupled. This is as opposed to the H^2 case where the two Riccati equations are independent of one another.

Referring to the estimation problem (1.7.26) shows that the estimator $\mathcal{K}_e$ estimates the output of the system, $\mathcal{L}_{12}\mathcal{L}_{22}^{-1}$, using noisy observations of the output of the system $\mathcal{P}_{21}\mathcal{L}_{22}^{-1}$ (see Sec. 1.2). However, we shall later see that in actually trying to find the H^∞ measurement feedback controller (say, for systems described by state-space models) it is much easier to keep the physical meaning of the separation in mind (such as identifying the signals a and b in the above proof), rather than to explicitly use the solution of Theorem 1.7.2.

Finally, it should also be noted that, in the measurement feedback case, the central solution has the maximum entropy property of

$$\max_{\text{causal } \mathcal{K}} \ \log\det\left[\gamma^2 I - \mathcal{T}_{\mathcal{K}}^{c*}\mathcal{T}_{\mathcal{K}}^{c}\right]. \tag{1.7.27}$$

1.7.3 Special Cases

The general solutions presented so far subsume both the finite and infinite horizon cases. We now briefly present these.

The Infinite Horizon Case

Let us first begin with the full information problem for which we have the following result.

Theorem 1.7.3 (Infinite Horizon H^∞ Full Information Controller) *(a) A causal controller, $K(z)$, that achieves*

$$\left\| \begin{bmatrix} P_1(z) + P_2(z)K(z) \\ K(z) \end{bmatrix} \right\|_\infty < \gamma,$$

exists if, and only if, there exists a dual canonical factorization of the form

$$\begin{gathered} \begin{bmatrix} I_{m_2} + P_2(z)P_2^*(z^{-*}) & P_2(z)P_1^*(z^{-*}) \\ P_1(z)P_2^*(z^{-*}) & -\gamma^2 I_{m_1} + P_2(z)P_2^*(z^{-*}) \end{bmatrix} = \\ \begin{bmatrix} L_{11}^*(z^{-*}) & L_{21}^*(z^{-*}) \\ L_{12}^*(z^{-*}) & L_{22}^*(z^{-*}) \end{bmatrix} \begin{bmatrix} I_{m_2} & 0 \\ 0 & -I_{m_1} \end{bmatrix} \begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}, \end{gathered} \tag{1.7.28}$$

with $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ *and $L_{11}(z)$ minimum phase and proper, and $L_{21}(z)$ causal and strictly proper. If this is the case, then all possible H^∞ full information controllers of level γ are given by*

$$K(z) = \left(L_{11}(z) - Q(z)L_{21}(z)\right)^{-1} \left(Q(z)L_{22}(z) - L_{12}\right)^{-1}, \tag{1.7.29}$$

where $Q(z)$ is any causal and strictly contractive transfer matrix, i.e., $Q(e^{j\omega})Q^*(e^{j\omega}) < I$, $\forall \omega \in [0, 2\pi]$. *The central filter results from the choice $Q(z) = 0$, so that*

$$K_{cen}(z) = -L_{11}^{-1}(z)L_{12}(z). \tag{1.7.30}$$

(b) Assume that $P_1(z)$ and $P_2(z)$ have state-space structure,

$$\begin{bmatrix} P_1(z) & P_2(z) \end{bmatrix} = L(zI - F)^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix},$$

with $\{F, G_2\}$ stabilizable and $\{F, L\}$ detectable. Then a solution to the full information H^∞ control problem with level γ exists if, and only if, there exists a solution to the (dual) DARE

$$P^c = F^* P^c F + L^* L - K_c^* R_c K_c, \tag{1.7.31}$$

with

$$K_c = R_c^{-1} \begin{bmatrix} G_2^* \\ G_1^* \end{bmatrix} P^c F \quad \text{and} \quad R_c = \begin{bmatrix} I_{m_2} & 0 \\ 0 & -\gamma^2 I_{m_1} \end{bmatrix} + \begin{bmatrix} G_2^* \\ G_1^* \end{bmatrix} P^c \begin{bmatrix} G_2 & G_1 \end{bmatrix} \tag{1.7.32}$$

such that

(i) $F_c \triangleq F - \begin{bmatrix} G_2 & G_1 \end{bmatrix} K_c$ *is stable.*

(ii) R_c *and* $\begin{bmatrix} I_{m_2} & 0 \\ 0 & -\gamma^2 I_{m_1} \end{bmatrix}$ *have the same inertia.*

If this is the case, then the $L_{ij}(z)$ in the dual canonical factorization (1.7.28) are given by

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \begin{bmatrix} (I_{m_2} + G_2^* P^c G_2)^{*/2} & (I_{m_2} + G_2^* P^c G_2)^{-1/2} G_2^* P^c G_1 \\ 0 & (\gamma^2 I_{m_1} - G_1^* P^c (I + G_2 G_2^* P^c)^{-1} G_1)^{*/2} \end{bmatrix} \times \left(\begin{bmatrix} I_{m_2} & 0 \\ 0 & I_{m_1} \end{bmatrix} + K_c (zI - F)^{-1} \begin{bmatrix} G_2 & G_1 \end{bmatrix} \right). \tag{1.7.33}$$

In particular, defining $R_{Gc} \triangleq I_{m_2} + G_2^* P^c G_2$, we have

$$\begin{cases} L_{11}(z) = R_{Gc}^{*/2} \left[I_{m_2} + R_{Gc}^{-1} G_2^* P^c F (zI - F)^{-1} G_2 \right] \\ L_{12}(z) = R_{Gc}^{*/2} \left[R_{Gc}^{-1} G_2^* P^c G_1 + R_{Gc}^{-1} G_2^* P^c F (zI - F)^{-1} G_1 \right] \end{cases}, \tag{1.7.34}$$

so that

$$K_{cen}(z) = -R_{Gc}^{-1} G_2^* P^c G_1 - R_{Gc}^{-1} G_2^* P^c F (zI - F_2)^{-1} (I - G_2 R_{Gc}^{-1} G_2^* P^c) G_1, \tag{1.7.35}$$

where $F_2 \triangleq F - G_2 R_{Gc}^{-1} G_2^* P^c F$.

Proof: The proof is the dual of the proof of Theorem 1.4.2 and will not be repeated here.

■

Remarks:

(i) It is straightforward to give state-space representations for the central controller (1.7.35). Indeed we have

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F_2 x_i + (I - G_2 R_{Gc}^{-1} G_2^* P^c) G_1 w_i \\ u_i & = & -R_{Gc}^{-1} G_2^* P^c F x_i - R_{Gc}^{-1} G_2^* P^c G_1 w_i \end{array} \right. , \tag{1.7.36}$$

where we have denoted the state variable by x_i, since it really is the state of the original state-space model. To see why, we can use the formulas $F_2 = F - G_2 R_{Gc}^{-1} G_2^* P^c F$ and $u_i = -R_{Gc}^{-1} G_2^* P^c F x_i - R_{Gc}^{-1} G_2^* P^c G_1 w_i$ in the above state equation to write,

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F x_i + G_1 w_i + G_2 u_i \\ u_i & = & -R_{Gc}^{-1} G_2^* P^c F x_i - R_{Gc}^{-1} G_2^* P^c G_1 w_i \end{array} \right. , \tag{1.7.37}$$

which is our desired result.

The above controller shows that the full information control signal is a function only of the current state x_i and the current exogenous input, w_i.[25] This is somewhat different from the state-feedback law that occurs in continuous-time H^∞ control. The reason is that we have insisted on causal controllers. Had we sought strictly causal controllers we would have obtained a state feedback law (for the central solution), as we now (very) briefly describe.

(ii) The solution to the problem of finding a *strictly* causal full information controller that achieves,

$$\left\| \begin{bmatrix} P_1(z) + P_2(z) K(z) \\ K(z) \end{bmatrix} \right\|_\infty < \gamma,$$

[25]The above statement is only true of the central controller. Other full information H^∞ controllers, as parametrized by (1.7.29), need not be so.

is exactly the same as the solution of Theorem 1.7.3 part (a) with the exception that, in the dual canonical factorization (1.7.28), instead of having $L_{21}(z)$ being strictly proper we require that $L_{12}(z)$ be strictly proper.

When we have state-space structure, the only difference in the existence condition is that condition (ii) should be replaced by[26]

$$-\gamma^2 I_{m_1} + G_1^* P^c G_1 < 0 \quad \text{and} \quad I_{m_2} + G_2^* P^c (I - \gamma^{-2} G_1 G_1^* P^c)^{-1} G_1 > 0. \quad (1.7.38)$$

In this case, the $L_{ij}(z)$ are given by,

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \begin{bmatrix} (I_{m_2} + G_2^* P^c (I - \gamma^{-2} G_1 G_1^* P^c)^{-1} G_1)^{*/2} & 0 \\ G_1^* P^c G_2 (\gamma^2 I_{m_1} - G_1^* P^c G_1)^{-1/2} & (\gamma^2 I_{m_1} - G_1^* P^c G_1)^{*/2} \end{bmatrix} \times \left(\begin{bmatrix} I_{m_2} & 0 \\ 0 & I_{m_1} \end{bmatrix} + K_c (zI - F)^{-1} \begin{bmatrix} G_2 & G_1 \end{bmatrix} \right). \quad (1.7.39)$$

The above expression allows us to find the central solution $K(z) = -L_{11}(z)^{-1} L_{12}(z)$, which turns out to be,

$$K_{cen}(z) = \tilde{K}_c (zI - F + G_2 \tilde{K}_c)^{-1} G_1, \quad (1.7.40)$$

where

$$\tilde{K}_c = (I_{m_2} + G_2^* \tilde{P}^c G_2)^{-1} G_2^* \tilde{P}^c F \quad \text{and} \quad \tilde{P}^c = P^c - P^c G_1 (-\gamma^2 I_{m_1} + G_1^* P^c G_1)^{-1} G_1^* P^c. \quad (1.7.41)$$

The above central solution can be shown to have the following state-space model,

$$\begin{cases} x_{i+1} &= F x_i + G_1 w_i + G_2 u_i \\ u_i &= \tilde{K}_c x_i \end{cases}, \quad (1.7.42)$$

which is, of course, a state-feedback control law.

(iii) Note, once more, the surprising similarities between the above full information H^∞ controller and the H^2 full information controller of Theorem 1.6.3.

[26]Note that this condition is more stringent than (ii). If it is true, then R_c and $I_{m_2} \oplus (-I_{m_1})$ have the same inertia, but not necessarily vice versa.

Let us now consider the measurement feedback problem[27] where the model is given by,

$$\left\{\begin{array}{lcl} s(z) & = & P_{11}(z)w(z) + P_{12}(z)u(z) \\ y(z) & = & P_{21}(z)w(z) + P_{22}(z)u(z) + v(z) \\ u(z) & = & K(z)y(z) \end{array}\right. . \tag{1.7.43}$$

Since the solution in this case is simply a restatement of Theorem 1.7.2 where one has to replace transfer operators by their corresponding transfer matrices, we shall not repeat it here. Therefore, we shall instead focus on the case where we have state-space structure,

$$\left[\begin{array}{cc} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{array}\right] = \left[\begin{array}{c} L \\ H \end{array}\right] (zI - F)^{-1} \left[\begin{array}{cc} G_1 & G_2 \end{array}\right], \tag{1.7.44}$$

and study the consequences of Theorem 1.7.2.

Recall from the proof (and separation principle) of Theorem 1.7.2 that the solution to the measurement feedback problem is given by that choice of controller that renders the mapping from the signals $\{b_i\}$ and $\{v_i\}$ to the signal $\{a_i\}$ to be a strict contraction, where

$$\left\{\begin{array}{lcl} a(z) & = & L_{11}(z)u(z) + L_{12}(z)w(z) \\ b(z) & = & L_{21}(z)u(z) + L_{22}(z)w(z) \end{array}\right. , \tag{1.7.45}$$

and where the $L_{ij}(z)$ are found from the dual canonical factorization,

$$\begin{array}{c} \left[\begin{array}{cc} I_{m_2} + P_{12}(z)P_{12}^*(z^{-*}) & P_{12}(z)P_{11}^*(z^{-*}) \\ P_{11}(z)P_{12}^*(z^{-*}) & -I_{m_1} + P_{12}(z)P_{12}^*(z^{-*}) \end{array}\right] = \\ \left[\begin{array}{cc} L_{11}^*(z^{-*}) & L_{21}^*(z^{-*}) \\ L_{12}^*(z^{-*}) & L_{22}^*(z^{-*}) \end{array}\right] \left[\begin{array}{cc} I_{m_2} & 0 \\ 0 & -I_{m_1} \end{array}\right] \left[\begin{array}{cc} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{array}\right], \end{array} \tag{1.7.46}$$

with $\left[\begin{array}{cc} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{array}\right]$ and $L_{11}(z)$ minimum phase and proper, and $L_{21}(z)$ causal and strictly proper.

Let us now identify the $L_{ij}(z)$ (and thereby the $a(z)$ and $b(z)$) using the state-space descriptions given. Using Eq. (1.7.33) in Theorem 1.7.3 we can, after some

[27] We have, once more, for simplicity, assumed that $\gamma = 1$.

algebra, write

$$\left\{\begin{array}{lcl} L_{11}(z) & = & R_{Gc}^{*/2} + R_{Gc}^{*/2} K_d (zI - F)^{-1} G_2 \\ L_{12}(z) & = & R_{Gc}^{-1/2} G_2^* P^c G_1 + R_{Gc}^{*/2} K_d (zI - F)^{-1} G_1 \\ L_{21}(z) & = & \Delta^{*/2} K_w (zI - F)^{-1} G_2 \\ L_{22}(z) & = & \Delta^{*/2} + \Delta^{*/2} K_w (zI - F)^{-1} G_2 \end{array}\right. , \tag{1.7.47}$$

where we have defined

$$K_d = R_{Gc}^{-1} G_2^* P^c F \quad , \quad K_w = \Delta^{-1} G_1^* P^c (I + G_2 G_2^* P^c)^{-1} F \tag{1.7.48}$$

and

$$R_{Gc} = I_{m_2} + G_2^* P^c G_2 > 0 \quad , \quad \Delta = I_{m_1} - G_1^* P^c (I + G_2 G_2^* P^c)^{-1} G_1 > 0 \tag{1.7.49}$$

and where P^c is given by the solution to the DARE (1.7.31) with the aforementioned properties. This allows us to write

$$a(z) = R_{Gc}^{*/2} K_d (zI - F)^{-1} \left[G_1 w(z) + G_2 u(z)\right] + R_{Gc}^{-1/2} G_2^* P^c G_1 w(z) + R_{Gc}^{*/2} u(z),$$

which has state-space description,

$$\left\{\begin{array}{rcl} x_{i+1} & = & F x_i + G_1 w_i + G_2 u_i \\ a_i & = & R_{Gc}^{*/2} K_d x_i + R_{Gc}^{-1/2} G_2^* P^c G_1 w_i + R_{Gc}^{*/2} u_i \end{array}\right. , \tag{1.7.50}$$

and

$$b(z) = \Delta^{*/2} w(z) + \Delta^{*/2} K_w (zI - F)^{-1} \left[G_1 u(z) + G_2 u(z)\right],$$

which has state-space description,

$$\left\{\begin{array}{rcl} x_{i+1} & = & F x_i + G_1 w_i + G_2 u_i \\ b_i & = & \Delta^{*/2} K_w x_i + \Delta^{*/2} w_i \end{array}\right. . \tag{1.7.51}$$

Now from the last of the above equations, we conclude that $w_i = \Delta^{-*/2} b_i - K_w x_i$. Inserting this expression into the state-space model (1.7.50) yields,

$$\left\{\begin{array}{rcl} x_{i+1} & = & (F - G_1 K_w) x_i + G_1 \Delta^{-*/2} b_i + G_2 u_i \\ a_i & = & (R_{Gc}^{*/2} K_d - R_{Gc}^{-1/2} G_2^* P^c G_1 K_w) x_i + R_{Gc}^{-1/2} G_2^* P^c G_1 \Delta^{-*/2} b_i + R_{Gc}^{*/2} u_i \end{array}\right. . \tag{1.7.52}$$

Since a_i is the signal that we are trying to make small (through our choice of control signal, u_i), we can define the desired signal,

$$d_i \triangleq -R_{Gc}^{*/2}K_u x_i - R_{Gc}^{-1/2}G_2^* P^c G_1 \Delta^{-*/2} b_i \quad , \quad K_u \triangleq (K_d - R_{Gc}^{-1}G_2^* P^c G_1 K_w) \quad (1.7.53)$$

and gather the equations that we have to write,

$$\left\{ \begin{array}{rcl} x_{i+1} & = & (F - G_1K_w)x_i + G_1\Delta^{-*/2}b_i + G_2u_i \\ d_i & = & -R_{Gc}^{*/2}K_u x_i - R_{Gc}^{-1/2}G_2^*P^cG_1\Delta^{-*/2}b_i \\ y_i & = & Hx_i + v_i \end{array} \right. . \quad (1.7.54)$$

Therefore the problem of finding a causal controller that makes the mapping from b and v to a strictly contractive is equivalent to the problem of constructing a causal estimate of d_i, such that the mapping from the disturbances $\{b_i, v_i\}$ to the estimation errors, $\{\tilde{d}_{i|i} \triangleq d_i - \hat{d}_{i|i}\}$, is strictly contractive. But this is now a standard estimation problem that we know how to solve.[28] Indeed, using a slight generalization of Theorem 1.4.2, the central solution is given by

$$R_{Gc}^{*/2}u_i = \hat{d}_{i|i} = -R_{Gc}^{*/2}K_u\hat{x}_{i|i}, \quad (1.7.55)$$

where $R_{Gc}^{*/2}K_u\hat{x}_{i|i}$ satisfies,

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1} & = & (F - G_1K_w - K_1H)\hat{x}_i + K_1y_i + G_2u_i \\ R_{Gc}^{*/2}K_u\hat{x}_{i|i} & = & R_{Gc}^{*/2}K_u(I - PH^*R_{He}^{-1}H)\hat{x}_i + R_{Gc}^{*/2}K_uPH^*R_{He}^{-1}y_i \end{array} \right. , \quad (1.7.56)$$

with $K_1 = (F - G_1K_w)PH^*R_{He}^{-1}$ and $R_{He} = I_p + HPH^*$. Moreover, P is that solution of the DARE,

$$P = (F - G_1K_w)P(F - G_1K_w)^* + G_1\Delta^{-1}G_1^* - K_pR_eK_p^*, \quad (1.7.57)$$

with

$$K_p = \left\{(F - G_1K_w)P \left[\begin{array}{cc} H^* & K_u^*R_{Gc}^{1/2} \end{array} \right] + \left[\begin{array}{cc} G_1\Delta^{-1}G_1^*P^cG_2R_{G^c}^{-*/2} & 0 \end{array} \right] \right\} R_e^{-1},$$

[28]In fact, it is a slight generalization of the standard problem we have been considering so far, since here the disturbance b_i also enters directly into the equation for the desired signal, $d_i = -R_{Gc}^{*/2}K_u x_i - R_{Gc}^{-1/2}G_2^*P^cG_1\Delta^{-*/2}b_i$. Nonetheless, the problem can still be solved using the same techniques and the only difference turns out to be a small modification in the K_p of the DARE.

and

$$R_e = \begin{bmatrix} I_p & 0 \\ 0 & -I_{m_2} \end{bmatrix} + \begin{bmatrix} H \\ R_{Gc}^{*/2} K_u \end{bmatrix} P \begin{bmatrix} H^* & K_u^* R_{Gc}^{1/2} \end{bmatrix},$$

that satisfies the following conditions,

(i) $F_p \triangleq F - G_1 K_w - K_p \begin{bmatrix} H \\ R_{Gc}^{*/2} K_u \end{bmatrix}$ is stable.

(ii) R_e and $I_p \oplus (-I_{m_2})$ have the same inertia.

Incidentally, from Theorem 1.4.2, the existence of a P with the above properties is necessary and sufficient for the existence of a solution to the measurement feedback problem.

Note also that replacing $u_i = -K_u(I - PH^* R_{He}^{-1} H)\hat{x}_i - K_u P H^* R_{He}^{-1} y_i$ into (1.7.56), we can write

$$\begin{cases} \hat{x}_{i+1} = F_m \hat{x}_i + (K_1 - G_2 K_u P H^* R_{He}^{-1}) y_i \\ u_i = K_u(I - PH^* R_{He}^{-1} H)\hat{x}_i - K_u P H^* R_{He}^{-1} y_i \end{cases} . \tag{1.7.58}$$

where $F_m \triangleq F - G_1 K_w - K_1 H - G_2 K_u (I - PH^* R_{He}^{-1} H)$, which is the desired system that maps the observations, y_i, to the control signal, u_i.

We can now gather the results obtained so far in the following theorem where for simplicity we have taken $\gamma = 1$. [We have also, without loss of generality, taken $Q = I$, $R = I$, $Q^c = I$ and $R^c = I$ since these weighting matrices can be absorbed into the $\{G_1, G_2, H, L\}$.]

Theorem 1.7.4 (Infinite-Horizon H^∞ Measurement Feedback Controller) *Consider the state-space model*

$$\begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} = \begin{bmatrix} L \\ H \end{bmatrix} (zI - F)^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix},$$

with $\{F, G_1\}$ and $\{F, G_2\}$ stabilizable and $\{F, H\}$ and $\{F, L\}$ detectable, and the transfer matrix,

$$T_K = \begin{bmatrix} P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21} & P_{12} K (I - P_{22} K)^{-1} \\ K (I - P_{22} K)^{-1} P_{21} & K (I - P_{22} K)^{-1} \end{bmatrix},$$

where for notational simplicity we have suppressed the dependence of the transfer matrices on z.

Then there exists a causal controller that achieves

$$\|T_K(z)\|_\infty < 1,$$

if, and only if, the DARE's,

$$\left\{\begin{array}{rcl} P^c & = & F^*P^cF + L^*L - K_c^*R_cK_c \\ P & = & (F-G_1K_w)P(F-G_1K_w)^* + G_1\Delta^{-1}G_1^* - K_pR_eK_p^* \end{array}\right. , \tag{1.7.59}$$

where

$$\left\{\begin{array}{rcl} K_c & = & R_c^{-1}\begin{bmatrix} G_2^* \\ G_1^* \end{bmatrix} P^cF \\ R_c & = & \begin{bmatrix} I_{m_2} & 0 \\ 0 & -I_{m_1} \end{bmatrix} + \begin{bmatrix} G_2^* \\ G_1^* \end{bmatrix} P^c \begin{bmatrix} G_2 & G_1 \end{bmatrix} \\ K_w & = & \Delta^{-1}G_1^*P^c(I+G_2G_2^*P^c)^{-1}F \\ \Delta & = & I_{m_1} - G_1^*P^c(I+G_2G_2^*P^c)^{-1}G_1 \\ K_p & = & \left\{(F-G_1K_w)P\begin{bmatrix} H^* & K_u^*R_{Gc}^{1/2} \end{bmatrix} + \begin{bmatrix} G_1\Delta^{-1}G_1^*P^cG_2R_{G^c}^{-*/2} & 0 \end{bmatrix}\right\} R_e^{-1} \\ R_e & = & \begin{bmatrix} I_p & 0 \\ 0 & -I_{m_2} \end{bmatrix} + \begin{bmatrix} H \\ R_{Gc}^{*/2}K_u \end{bmatrix} P \begin{bmatrix} H^* & K_u^*R_{Gc}^{1/2} \end{bmatrix} \\ \begin{bmatrix} K_w \\ K_u \end{bmatrix} & = & K_c \\ R_{Gc} & = & I_{m_2} + G_2^*P^cG_2 \end{array}\right. , \tag{1.7.60}$$

have solutions, P^c and P, such that

(i) $F_c \triangleq F - \begin{bmatrix} G_2 & G_1 \end{bmatrix} K_c$ *is stable.*

(ii) R_c *and* $I_{m_2} \oplus (-I_{m_1})$ *have the same inertia.*

(iii) $F_p \triangleq F - G_1K_w - K_p\begin{bmatrix} H \\ R_{Gc}^{*/2}K_u \end{bmatrix}$ *is stable.*

(iv) R_e *and* $I_p \oplus (-I_{m_2})$ *have the same inertia.*

If this is the case, then the "so-called" central controller is given by

$$K_{cen}(z) = -K_u P H^* R_{He}^{-1} - K_u(I - PH^* R_{He}^{-1} H)(zI - F_m)^{-1}(K_1 - G_2 K_u P H^* R_{He}^{-1}), \tag{1.7.61}$$

where $F_m = F - G_1 K_w - K_1 H - G_2 K_u(I - PH^* R_{He}^{-1} H)$, *which has the following state-space model,*

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1} & = & (F - G_1 K_w - K_1 H)\hat{x}_i + K_1 y_i + G_2 u_i \\ u_i & = & -K_u(I - PH^* R_{He}^{-1} H)\hat{x}_i - K_u P H^* R_{He}^{-1} y_i \end{array} \right. . \tag{1.7.62}$$

Remarks:

(i) Note that, unlike the H^2 measurement feedback control problem of Theorem 1.6.4, the Riccati equations for P^c and P are coupled. Indeed, the DARE for P depends on the solution of the DARE for P^c (but not vice versa). Therefore the separation in measurement feedback H^∞ control is not complete, *i.e.*, the estimation problem depends on the full information controller.

Through a suitable change of variables (essentially a bilinear transformation involving P and P^c) it is possible to come up with an auxiliary variable, say P^d, that satisfies a DARE which is independent of P^c. However, the price we have to pay is a certain coupling condition (on the spectral radius of $P^c P^d$) that has to be added to the existence conditions (i)-(iv). In fact, the first solutions to the measurement feedback H^∞ and risk-sensitive control problems (see [DGKF89] and [Whi90]) presented such Riccati equations and a corresponding separation principle. However, we believe that the separation principle given here is more natural,[29] and shall therefore not go into the details of defining P^d and deriving its DARE (which requires lengthy algebraic manipulations, anyway).

(ii) The solutions of the various H^∞ control and estimation problems presented so far require finding the *stabilizing* solution of a DARE and checking whether the solution has certain inertia properties. It turns out that there are many efficient

[29] In fact, the separation principle given here does not need to appeal to state-space models.

methods for both checking the existence of, and finding stabilizing solutions to, DARE's (see *e.g.*, [BLW91], [Meh91], [LR95] and the references therein). We shall say more about this in Chapter 8.

(iii) Proceeding in a similar vein it is also possible to derive measurement feedback H^∞ controllers that are strictly causal, however, for lack of space we shall not do so here.

State-Space Models

Let us finally, consider the finite horizon case where we have (possibly time-varying) state-space models which, for the full information problem, we may write as

$$\left\{\begin{array}{rcl} x_{i+1} & = & F_i x_i + G_{1,i} w_i + G_{2,1} u_i \\ s_i & = & L_i x_i \end{array}\right. , x_0, \quad 0 \le i \le N \tag{1.7.63}$$

and, for the measurement feedback problem,

$$\left\{\begin{array}{rcl} x_{i+1} & = & F_i x_i + G_{1,i} w_i + G_{2,1} u_i \\ s_i & = & L_i x_i \\ y_i & = & H_i x_i + v_i \end{array}\right. , x_0, \quad 0 \le i \le N. \tag{1.7.64}$$

Moreover, assume that the cost associated with the disturbances, x_0, $\{u_i\}$ and $\{v_i\}$ is given by

$$x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} w_i^* Q_i^{-1} w_i + \sum_{i=0}^{N} v_i^* R_i^{-1} u_i, \tag{1.7.65}$$

and that the objective cost function is

$$\sum_{i=0}^{N} s_i^* R_i^c s_i + \sum_{i=0}^{N} u_i^* Q_i^c u_i + x_{N+1}^* P_{N+1}^c x_{N+1}, \tag{1.7.66}$$

where $\Pi_0 > 0$, $Q_i > 0$, $R_i > 0$, $R_i^c \ge 0$, $Q_i^c \ge 0$ and $P_{N+1}^c \ge 0$ are given weighting matrices.

We shall not give the finite horizon full information and measurement feedback controllers here since they are very similar to the solutions given by Theorems 1.7.3

and 1.7.4. Essentially, the only difference is that the DARE's for P and P^c are replaced by Riccati recursions for the time-varying matrices, P_i and P_i^c, *i.e.*,

$$\left\{\begin{array}{rcl} P_i^c &=& F_i^* P_{i+1}^c F_i + L_i^* L_i - K_{c,i}^* R_{c,i} K_{c,i} \\ P_{i+1} &=& (F_i - G_{1,i}K_{w,i})P_i(F_i - G_{1,i}K_{w,i})^* + G_{1,i}\Delta_i^{-1}G_{1,i}^* - K_{e,i}R_{e,i}K_{p,i}^* \end{array}\right. , \tag{1.7.67}$$

(initialized with P_{N+1}^c and $P_0 = (\Pi_0^{-1} - P_0^c)^{-1}$) where

$$\left\{\begin{array}{rcl} K_{c,i} &=& R_{c,i}^{-1}\begin{bmatrix} G_{2,i}^* \\ G_{1,i}^* \end{bmatrix} P_{i+1}^c F_i \\ R_{c,i} &=& \begin{bmatrix} I_{m_2} & 0 \\ 0 & -I_{m_1} \end{bmatrix} + \begin{bmatrix} G_{2,i}^* \\ G_{1,i}^* \end{bmatrix} P_{i+1}^c \begin{bmatrix} G_{2,i} & G_{1,i} \end{bmatrix} \\ K_{w,i} &=& \Delta_i^{-1} G_{1,i}^* P_{i+1}^c (I + G_{2,i}G_{2,i}^* P_{i+1}^c)^{-1} F_i \\ \Delta_i &=& I_{m_1} - G_{1,i}^* P_{i+1}^c (I + G_{2,i}G_{2,i}^* P_{i+1}^c)^{-1} G_{1,i} \\ K_{p,i} &=& \left\{ (F_i - G_{1,i}K_{w_i})P_i \begin{bmatrix} H_i^* & K_{u,i}^* R_{Gc,i}^{1/2} \end{bmatrix} + \begin{bmatrix} G_{1,i}\Delta_i^{-1}G_{1,i}^* P_{i+1}^c G_{2,i} R_{G^c,i}^{-*/2} & 0 \end{bmatrix} \right\} R_{e,i}^{-1} \\ R_{e,i} &=& \begin{bmatrix} I_p & 0 \\ 0 & -I_{m_2} \end{bmatrix} + \begin{bmatrix} H_i \\ R_{Gc,i}^{*/2} K_{u,i} \end{bmatrix} P_i \begin{bmatrix} H_i^* & K_{u,i}^* R_{Gc,i}^{1/2} \end{bmatrix} \\ \begin{bmatrix} K_{w,i} \\ K_{u,i} \end{bmatrix} &=& K_c \\ R_{Gc,i} &=& I_{m_2} + G_{2,i}^* P_{i+1}^c G_{2,i} \end{array}\right. , \tag{1.7.68}$$

and that the stability requirements of Theorems 1.7.3 and 1.7.4 are dropped.

We end this section by noting that we will return to control problems, albeit through a slightly different approach, in Chapter 6. The relationships between the above Riccati recursions and the problem of recursively performing canonical and dual canonical factorizations will be taken up starting in Chapter 2. Finally, the asymptotic behaviour of the above controllers (in the time-invariant case) will be studied in Chapter 8.

1.8 Other Approaches to Estimation and Control

In Secs. 1.3 to 1.7 we gave a rather detailed overview of the H^2 and H^∞ approaches to estimation and control. It should be mentioned that, although it is fair to say that these are the two methodologies that have received the most attention in the literature, they are by no means the only approaches to these problems. We have already mentioned, in passing, the l_1 approach to robust estimation and control [DP87, DDB95], and should mention the various "mixed" strategy approaches, such as mixed H^2/H^∞ (which we shall explain in Chapter 12) [BH89, YBC92, KR92, LA94, ZGBD94a, ZGBD94b, Meg94, FFT94, FFL95, HHK96], mixed l_1/H^∞ [Rot95], mixed H^2/l_1 [Vou95], and other variations, as well as such methods as "unknown but bounded" estimation and control [Sch73b]. Of course, there are also other general approaches to estimation and control, such as adaptive control [AW89, SB89, KKK95, WW96], nonlinear control [Lef65, BMS88, Kha92, Vid93], and convex-optimization-based control [BB91, BGFB94], which are far beyond the scope of this thesis.

However, there are two other recent approaches to estimation and control that immediately lend themselves to the methodology presented in this thesis. These two are the so-called risk-sensitive (or LEQG) and game-theoretic approaches to filtering and control.

Risk-sensitive Control and Quadratic Game Theory

In the risk-sensitive framework, instead of choosing estimators and controllers that minimize the expected value of a certain quadratic cost function, as the H^2 framework suggests, attempt is made to choose estimators and controllers that minimize the expected value of an exponential-quadratic cost function [Jac73, SDJ74, Whi90, SFB92]. The ensuing theory is sometimes called LEQG (linear-exponential-quadratic-Gaussian) theory to reflect the facts that the resulting optimal estimators (and controllers) are linear, the cost function is the exponential of a quadratic, and the disturbances are assumed to be Gaussian random variables. The reason why it is also (following Whittle) called risk-sensitive estimation and control is that the criterion is risk-sensitive, in the sense that it depends on a real parameter that determines whether more or less weight should be given to higher or smaller errors. [Roughly speaking when more weight is given to smaller errors the criterion is risk-seeking, and when more weight is given to large errors the criterion is risk-averse.]

Following some pioneering work in game theory (motivated primarily by the field of economics [NM44, KT50]), since the mid 1960's there has been considerable interest in applying game-theoretic ideas and methods to estimation and control. In this framework, the problem of estimation (or control) is treated as a noncooperative two-player game, with one player (the opponent) being the exogenous signals, and the other player being the estimator (or controller). This approach, which treats the exogenous signals as malignant disturbances that compete against the estimator (or controller), is, of course, fundamentally different from the H^2 (or risk-sensitive) approach where the exogenous signals are simply taken to be random variables with known probability distributions. The class of games most often applied to estimation and control is the class of differential games with quadratic payoff [Isa65, Ber64, BH69], which, in the discrete-time case of interest to us, shall henceforth be referred to as a quadratic dynamic game.

An interesting aspect of the risk-sensitive and quadratic dynamic game approaches is that the resulting solutions bear many similarities to the solutions obtained via the H^2 and H^∞ approaches — in fact, estimators have an observer structure, full information controllers have state-feedback structure, the various observer and state-feedback gains are found from the solution of certain Riccati equations, etc.

The main reason for the above similarities, and indeed a major claim of this thesis, is that all the above problems can be related to stationarizing certain indefinite quadratic forms. In other words, these (seemingly different) problems are all unified (or can be treated in a unified fashion) since they can be related to certain *indefinite* quadratic forms from which a Krein space optimization problem (as developed in Chapter 2) can be inferred. We shall give the details of why this is the case in Chapters 3 and 4, but, for the time being, we can outline how this comes about in the following (somewhat pictorial) fashion.

(i) H^∞ problems:

$$\frac{\|\cdot\|^2}{\|\cdot\|^2} < \gamma^2$$

$$\Rightarrow \quad \|\cdot\|^2 - \gamma^{-2}\|\cdot\|^2 > 0$$

(ii) Risk-sensitive problems:

$$\min \; Ee^{\|\cdot\|^2}$$

$$\Rightarrow \quad \min \int e^{\|\cdot\|^2} e^{-x^* R_x^{-1} x} dx = \min e^{\min_x \left[\|\cdot\|^2 - x^* R_x^{-1} x\right]}$$

(iii) Quadratic game problems:

$$\min_u \max_w J(u, w, y), \quad J \;\; \text{quadratic and indefinite}$$

Therefore, starting from the next chapter, we shall develop a general theory for linear estimation in Krein spaces. This theory will then be applied to H^∞, risk-sensitive and quadratic dynamic game, estimation and control in Chapters 3, 4 and 6. This approach also has the major bonus of allowing for new results in these fields, as demonstrated throughout the thesis, and, in particular, as shown in Chapters 5, 7, 8 and 9.

Adaptive Filtering

Finally, a few remarks are in order for the fields of adaptive filtering, adaptive signal processing and adaptive neural networks. Adaptive filtering has been widely used (since the late 1950's) to cope with time-variations of system parameters and lack of a priori knowledge of the statistical properties of the input data [WS85, Hay96]. This puts it in contrast to Wiener and Kalman filter theory which require a priori statistical information. Indeed it was claimed that adaptive filtering algorithms "learn" the signal statistics as the signals are observed, and hence "adapt" to the inputs that they are presented with.

Although this may very well be the case, it was also claimed that adaptive filtering is therefore a field different from estimation theory, since estimation theory relies on a priori statistical information. This, of course, is based on too narrow a definition of estimation theory, since recent approaches in this field, such as the H^∞, l_1 and game-theoretic approaches, do not require any a priori statistical knowledge and are therefore fully compatible with the objectives of adaptive filtering.

Therefore it is fair to say that adaptive filtering is a subfield of estimation theory. Moreover, insofar as adaptive filtering algorithms are concerned, once these algorithms satisfy a certain optimality criterion (as does, say, the RLS algorithm, which is least-squares or H^2-optimal) then they are certainly a part of estimation theory, and once they do not satisfy such a criterion, they can be regarded, to some extent or the other, as ad hoc. In fact, in Chapter 9 we shall show that the celebrated LMS adaptive algorithm [WH60] (along with related instantaneous-gradient-based methods), which had long been regarded as an intuitively appealing method with many successful practical applications, but little theoretical justification, is indeed H^∞ optimal. This surprising fact demonstrates the connections between adaptive filtering and H^∞ estimation, which are further pursued in Chapter 11.

1.9 Scope and Contributions of Thesis

In this last section (of this introductory chapter) we shall present a brief overview of the scope and contributions of this thesis. For ease of reference, and to illustrate some of the pedagogical continuity between successive chapters, we shall survey the scope and contributions on a chapter by chapter basis. Browsing through the opening paragraphs of each chapter should also serve as a useful survey.

Chapter 2

Chapter 2 develops a self-contained theory for linear estimation in Krein spaces. A first motivation for the study of indefinite metric spaces was noted in Secs. 1.3 to 1.7 where we saw that the solution of various estimation and control problems could be given in terms of the canonical factorizations of certain indefinite transfer operators. Chapter 2 begins with a further motivation of indefinite metric spaces, via the celebrated Kalman-Yakubovich-Popov (KYP) lemma, and then proceeds to study Krein spaces and to compare their geometrical properties (such as the existence of projections onto linear subspaces) with the corresponding geometrical properties of Hilbert (or definite metric) spaces. It turns out that while Hilbert spaces and

Krein spaces share many characteristics, they differ in special ways that mark out the differences between the conventional H^2 theories and the more recent H^∞ theories.

We then relate Krein space projections to the computation of stationary points of certain indefinite quadratic forms, and give conditions as to when these stationary points correspond to (stochastic or deterministic) minima. The results are then specialized to state-space models, more specifically, to linear state-space models driven by inputs that lie in a Krein space. This then leads to the centerpiece of the ensuing theory, viz., a Krein space generalization of the classical Kalman filter. This (so-called) Krein space Kalman filter allows one to recursively compute the stationary point of certain indefinite quadratic forms, or to recursively compute the canonical factorization of certain indefinite Gramians (or transfer operators), and therefore plays a crucial role in the solution of problems in several areas such as H^2 and H^∞ estimation and control, quadratic game theory, risk-sensitive optimization and adaptive filtering.

The results presented in this chapter serve as a backbone for the remainder of the thesis and will be repeatedly used throughout the remaining chapters. Indeed it will be shown that the Krein space formalism developed in Chapter 2 serves as a unified approach to treating problems in H^2, H^∞, risk-sensitive and game-theoretic estimation, control and adaptive filtering, and allows one to extend to these new settings many of the results developed and obtained in H^2 theory over the last three to four decades.

The appendix of Chapter 2 presents a new stochastic interpretation of the KYP lemma and constructs a simple proof based on Krein space geometry.

Chapter 3

Chapter 3 deals with the study and solution of H^∞ filtering problems, using the Krein space estimation theory developed in Chapter 2. The basic approach is to associate an indefinite quadratic form with the H^∞ filtering problem, and to use the Krein space Kalman filter to compute the stationary point of the quadratic form and to check its conditions for a minimum. The major result is that H^∞ filters are nothing more than certain Krein space Kalman filters — thus explaining the surprising similarities between H^∞ filters and conventional Kalman filters. The chapter solves the so-called H^∞ a posteriori, a priori and smoothed estimation problems and parametrizes all possible solutions. Although most of the results on H^∞ filtering, apart from, say, certain equivalent conditions for the existence of solutions, are not new, the approach and the derivations are. [Incidentally, we believe that the approach presented here is the most natural.] The chapter also solves the l-step ahead H^∞ prediction problem which appears to be new — we are not aware of solutions to this problem in the literature.

Chapter 4

Chapter 3 describes some further (filtering) applications of the Krein space estimation theory of Chapter 2. Specifically, these applications are risk-sensitive filtering, quadratic game-theoretic filtering, and finite-memory adaptive filtering. The major point is that all these problems can be cast into the problem of calculating the stationary point of certain indefinite quadratic forms, and that by considering the appropriate state space models and error Gramians, we can use the Krein space estimation theory to calculate these stationary points and study their properties. Although many of the connections between H^∞, risk-sensitive and quadratic game-theoretic estimation and control are known in the literature, the material of this chapter sheds further light on these connections and provides for a new perspective. Moreover, while the solution to finite-memory (or so-called sliding window) adaptive filtering problems are wellknown in the literature, the development presented here, and the connections with Kalman filtering, are new. Finally, some connections to new work on suboptimal recursive total least-squares algorithms are mentioned.

Chapter 5

We have repeatedly claimed that the major bonus of the Krein space approach to H^∞, game-theoretic, and risk-sensitive, estimation and control is that, apart from rather more transparent derivations of existing results (as done in Chapters 3 and 4), it shows a way to apply to the H^∞ (and these other) settings many of the results developed for Kalman filtering and LQG control over the last three decades. Chapter 5 is the first place where we truly deliver on this claim by developing square-root array algorithms and Chandrasekhar recursions for H^∞ filtering problems. These are the generalizations of the (numerically superior) conventional square-root arrays and (fast) Chandrasekhar recursions to the Krein space setting. The H^∞ square-root algorithms involve propagating the indefinite square-root of the quantities of interest and have the property that the appropriate inertia of these quantities is preserved. For systems that are constant, or whose time-variation is structured in a certain way, the Chandrasekhar recursions allow a reduction in the computational effort per iteration from $O(n^3)$ to $O(n^2)$, where n is the number of states. The H^∞ square-root and Chandrasekhar recursions both have the interesting feature that one does not need to explicitly check for the inertia conditions required for the existence of H^∞ filters. These conditions are built into the algorithms themselves so that an H^∞ estimator of the desired level exists if, and only if, the algorithms can be executed. [All the results of this chapter are new.]

Chapter 6

Chapter 6 deals with duality in (definite and indefinite metric) linear spaces. Although not the conventional way for introducing duality in systems and control theory, duality here is introduced through the geometrical notion of dual bases for linear spaces spanned by a set of nonorthogonal basis vectors. Apart from having conceptual value, the resulting duality is a useful tool in the study of various problems of interest, and, in particular, in allowing certain a dual approach to (definite and indefinite) quadratic problems. A large part of this chapter is devoted to the use of duality in (H^2, H^∞, game-theoretic and risk-sensitive) control. Although the control results given in this chapter are already available in the literature, the derivations are new, and effectively combine the use of duality (here duality with estimation problems) and the Krein space theory of Chapter 2. All results are based on a certain indefinite LQR (linear-quadratic-regulator) problem from which the H^2, H^∞, game-theoretic and risk-sensitive solutions follow as special cases.

Chapter 7

Chapter 7 studies the celebrated discrete-time algebraic Riccati equation (DARE) which arises in an impressive range of applications in systems and control theory. Although a great deal is known about the Riccati equation when the coefficient matrices are positive semi-definite, much less is known when these coefficients are indefinite matrices. In this chapter the DARE is considered in the full generality of this, so-called, indefinite case and the results are then particularized to some important special cases (essentially the special cases that arise in H^2 and H^∞ estimation and control). The approach taken in this chapter is through the introduction of a certain (so-called) Popov function whose factorizations are intimately related to solutions of the DARE.[30] The main result is that solutions to the DARE, or more more precisely a system of discrete-time algebraic Riccati equations (SDARE), exists if, and only if, a certain proper factorization of the Popov function exists. Additional conditions are then given under which the solution to the DARE becomes stabilizing, Hermitian, positive semi-definite, etc. The DARE is also related to a so-called Hamiltonian matrix, from which the famous invariant subspace method can be obtained that actually computes solutions to the DARE. Some examples are also included to illustrate the significance of the results.

Chapter 8

Chapter 8 focuses on the behaviour of the Riccati recursion with time-invariant coefficient matrices, as time progresses to infinity. The main objective is to find conditions under which, for a given initial condition, the solution to the Riccati recursion converges to a solution of the associated DARE. The main result states that if, at each time instant, a certain inertia condition is met, then the Riccati recursion (exponentially) converges to the unique stabilizing solution (assuming such a solution exists) of the associated DARE. In the general case, the aforementioned inertia conditions need to be recursively checked, however, in some special cases they may be reduced

[30] The Popov function can be regarded as the generalization of the usual power spectral density function to indefinite metric spaces.

to more simple and more explicit requirements on the initial condition. In particular, when the coefficient matrices of the Riccati recursion are positive semi-definite, convergence of the Riccati recursion can be guaranteed for some indefinite, and even negative semi-definite, initial conditions (provided they are bounded below by a certain negative semi-definite matrix). Moreover, in the case frequently encountered in H^∞ filtering and control, convergence is guaranteed for all positive semi-definite initial conditions that are less than or equal to the unique positive semi-definite solution of a related Lyapunov equation. [We believes that Chapter 8 furnishes a very direct approach to establishing the convergence of the Riccati recursion. Moreover, to the best of our knowledge, the results obtained here are more general than those to have appeared in the literature and subsume, as special cases, all of the earlier given results.]

Chapter 9

Chapter 9 uses the connection between adaptive filtering and state-space estimation to study adaptive filtering with an H^∞ criterion. In particular, it is shown that the celebrated LMS (least-mean-squares) adaptive algorithm is H^∞ optimal. The LMS algorithm has been long regarded as an approximate solution to either a stochastic or a deterministic least-squares problem, and it essentially amounts to updating the weight vector estimates along the direction of the instantaneous gradient of a quadratic cost function. In this chapter it is shown that LMS can be regarded as the exact solution to a minimization problem in its own right. Namely, it is established that it is a minimax filter: it minimizes the maximum energy gain from the disturbances to the prediction errors, while the closely related so-called normalized LMS algorithm minimizes the maximum energy gain from the disturbances to the filtered errors. Moreover, since these algorithms are central H^∞ filters, they minimize a certain exponential cost function and are thus also risk-sensitive optimal. The various implications of these results are also discussed, and it is shown how they provide theoretical justification for the widely observed excellent robustness properties of the LMS filter.

Chapter 10

In order to compare the robustness of other adaptive filtering algorithms with the (H^∞-optimal) LMS and normalized LMS algorithms, Chapter 10 studies the robustness of least-squares-based adaptive filters, such as the RLS algorithm,from the H^∞ point of view. The basic result is the derivation of certain upper and lower bounds for the H^∞ norm of the RLS algorithm (in fact, more generally, of the Kalman filter) with respect to prediction and filtered errors. The main conclusion is that, unlike LMS and normalized LMS which do not allow for any amplification of the disturbances, the RLS algorithm does allow for such amplification. This fact can be especially pronounced in the prediction error case. Moreover, it is also shown that the H^∞ norm for RLS is data-dependent, whereas for LMS and normalized LMS it was not so. [The H^∞ norm was simply unity.] The significance of the results are also discussed.

Chapter 11

The results of Chapters 9 and 10 indicate that there may be great promise in the interplay of adaptive filtering and H^∞ estimation theory. To continue with this approach and line of reasoning, Chapter 11 presents a preliminary study of the design of adaptive filters using the H^∞ criterion. The strength of H^∞-optimal adaptive filters lies in the fact that they guarantee the smallest possible estimation error energy over all possible disturbances of fixed energy, and are therefore robust with respect to model uncertainties and lack of statistical information on the exogenous signals. Specifically, this chapter studies the problem of prediction of the weight vector itself, and for the purpose of coping with time-variations, exponentially weighted, finite-memory and time-varying adaptive filtering. This results in some new adaptive filtering algorithms that may be useful in uncertain and non-stationary environments. The presentation of the chapter is brief and the major goal is to only demonstrate some of the possibilities.

Chapter 12

The final chapter concludes with some brief remarks on various directions for future research that are suggested by the methods and results presented in this thesis. In particular, it introduces and motivates the mixed H^2/H^∞ approach to estimation and control.