# Chapter 4

# Further Applications

In this chapter we show that several interesting problems in risk-sensitive estimation and control, dynamic quadratic game theory, and finite memory adaptive filtering follow as special cases of the Krein space linear estimation theory developed in Chapter 2. The major point is that all these problems can be cast into the problem of calculating the stationary point of certain indefinite quadratic forms, and that by considering the appropriate state space models and error Gramians, we can use the Krein space estimation theory to calculate these stationary points and study their properties. The approach discussed here allows for interesting generalizations, such as finite memory adaptive filtering with varying sliding patterns and suboptimal recursive total least-squares algorithms.

## 4.1 Introduction

The classical Kalman filter can be viewed as a recursive procedure that minimizes the expected value of a certain quadratic cost function. Recently there has also been increasing interest in an alternative so-called exponential-quadratic cost function [Jac73, SDJ74, Whi90, SFB92], and estimators (and controllers) that minimize its expected value. The ensuing theory is sometimes called LEQG (linear-exponential-quadratic-Gaussian) theory to reflect the facts that the resulting optimal estimators (and controllers) are linear, the cost function is the exponential of a quadratic, and the disturbances are assumed to be Gaussian random variables. It is also called (following Whittle) risk-sensitive estimation and control, since the criterion is risk-sensitive, in the sense that it depends on a real parameter that determines whether more or less weight should be given to higher or smaller errors. [Roughly speaking when more weight is given to smaller errors the criterion is risk-seeking, and when more weight is given to large errors the criterion is risk-averse.] The filters obtained within this framework are termed risk-sensitive and include the conventional Kalman filter as a (so-called risk-neutral) special case.

Following some pioneering work in game theory (motivated primarily by the field of economics [NM44, KT50]), since the mid 1960's there has been considerable interest in applying game-theoretic ideas and methods to estimation and control. In this framework, the problem of estimation (or control) is treated as a noncooperative two-player game, with one player (the opponent) being the exogenous signals, and the other player being the estimator (or controller). This approach, which treats the exogenous signals as malignant disturbances that compete against the estimator (or controller), is, of course, fundamentally different from the $H^2$ (or risk-sensitive) approach where the exogenous signals are simply taken to be random variables with known probability distributions. The class of games most often applied to estimation and control is the class of differential games with quadratic payoff [Isa65, Ber64, BH69], which, in the discrete-time case of interest to us, shall henceforth be referred to as a quadratic dynamic game. One reason for its pervasive use may be that the solutions to quadratic dynamic games bear many similarities to the solutions of $H^2$ estimation and control — estimators have an observer structure, full information controllers have state-feedback structure, the various observer and state-feedback gains are found from the solution of certain Riccati equations, etc.

In fact, it has recently been shown that there is a close connection between $H^\infty$ estimation and control, game theory, and risk-sensitive estimation and control [GD88, Bas89, LAKG92]. Indeed, it turns out the central $H^\infty$ estimators and controllers, as well as the risk-sensitive optimal estimators and controllers can be derived as solutions to a certain quadratic dynamic game. [This observation is the driving force of the game-theoretic approach to $H^\infty$ control.]

In this chapter we shall shed further light on the connections between these theories by using the Krein space approach of Chapter 2. Indeed, we shall see that, as with $H^\infty$ estimation and control, risk-sensitive estimation and control problems, quadratic games, and finite memory adaptive filtering problems lead almost by inspection to indefinite deterministic quadratic forms. Following Chapters 2 and 3, we can solve these problems by constructing the corresponding Krein space models. Once this is done, the Krein space Kalman filter solutions can be written down immediately, and the existence conditions for the problem can also be expressed in terms of quantities easily related to the basic Riccati equations of the Kalman filter.

The remainder of the chapter is organized as follows. In Section 4.2 we describe the problem of risk-sensitive estimation, and show that a risk-sensitive estimator is one that computes the stationary point of a certain (possibly indefinite) quadratic form, provided that this quadratic form has a minimum over a certain set of variables. By considering a corresponding Krein state space model, we use the results of Chapter 2 to derive conditions for the existence of the minimum, and to show that the Krein space projection also solves the risk-sensitive estimation problem. We then derive risk-sensitive a posteriori, a priori, smoothed, and $l$-step ahead filters parallel to what was done in Chapter 3. Sec. 4.3 gives a very brief introduction to the theory of quadratic dynamic games. We show that such games lend themselves to analysis by Krein space methods since the cost (or payoff) function in is an indefinite quadratic form. Moreover, we show that the (max-min, min-max, or saddle point) solutions to such games are found by stationarizing these quadratic forms and checking for certain minimizing and maximizing conditions. Some specific examples from state-space estimation and control are also provided. Finite memory adaptive filtering is studied in Sec. 4.4, where the Krein space approach is used to solve this problem, and to connect it to state-space approaches to adaptive filtering. An interesting byproduct of our analysis is the physical interpretation of innovations with negative Gramian as corresponding to the loss of information. The chapter is concluded in Sec. 4.5 with some remarks on further applications of the Krein space framework, *e.g.*, to recursive solutions of (suboptimal) total least-squares problems.

## 4.2 Risk-Sensitive Estimation

The so-called risk-sensitive (or exponential cost) criterion was introduced in [Jac73] and further studied in [SDJ74, Whi90, SFB92]. Glover and Doyle [GD88] noticed their close connection to the $H^\infty$ filters of Chapter 3. We shall make this connection in a different way by introducing an appropriate quadratic form.

### 4.2.1 The Exponential Cost Function

We again start with a state-space model of the form

$$\begin{cases} \mathbf{x}_{j+1} = F_j\mathbf{x}_j + G_j\mathbf{u}_j, & j \geq 0 \\ \mathbf{y}_j = H_j\mathbf{x}_j + \mathbf{v}_j \end{cases} \tag{4.2.1}$$

However, we now assume that the initial condition, $\mathbf{x}_0$, and the driving and measurement disturbances, $\{\mathbf{u}_j\}$, and $\{\mathbf{v}_j\}$ are independent zero mean Gaussian random variables with covariances with

$$E\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}\begin{bmatrix} \mathbf{x}_0^* & \mathbf{u}_j^* & \mathbf{v}_j^* \end{bmatrix} = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & 0 \\ 0 & 0 & R_i\delta_{ij} \end{bmatrix}. \tag{4.2.2}$$

As we have seen in Sec. 1.3, conventional $H^2$ estimators, such as the Kalman filter, estimate the quantity $\mathbf{s}_j = L_j\mathbf{x}_j$ from the observations $\{\mathbf{y}_k\}_{k=0}^m$ (*i.e.*, $\check{s}_{j|m} = \mathcal{F}_i(\mathbf{y}_0, \ldots, \mathbf{y}_m)$) by performing the following minimization

$$\min_{\{\check{\mathbf{s}}_{j|m}\}} E\left[\mathbf{C}_i\right] \tag{4.2.3}$$

where $\mathbf{C}_i \triangleq \sum_{j=0}^{i}(\check{\mathbf{s}}_{j|m} - L_j\mathbf{x}_j)^*(\check{\mathbf{s}}_{j|m} - L_j\mathbf{x}_j)$ and $E\left[\cdot\right]$ denotes expectation. As we have seen earlier, the choices $m = j$, $m = j-1$, $m = i$, and $m = j-l$ correspond to the a posteriori, the a priori, the smoothed, and the $l$-step ahead, estimation problems, respectively. Moreover, the expectation is taken over the jointly Gaussian random variables $\{\mathbf{x}_0, \{\mathbf{u}_j\}_{j=0}^i, \{\mathbf{y}_j\}_{j=0}^i\}$.[1] Using the fact that $\mathbf{x}_0$ and the $\{\mathbf{u}_j\}$ and $\{\mathbf{v}_j\}$ are independent Gaussian random variables with covariances given by (4.2.2), and the

[1] Note that this set of random variables is equivalent to the set $\{\mathbf{x}_0, \{\mathbf{u}_j\}_{j=0}^i, \{\mathbf{v}_j\}_{j=0}^i\}$.

fact that $\mathbf{v}_j = \mathbf{y}_j - H_j\mathbf{x}_j$, allows us to write the joint probability distribution of the $\{\mathbf{x}_0, \{\mathbf{u}_j\}_{j=0}^i, \{\mathbf{y}_j\}_{j=0}^i\}$ as

$$p(x_0; u_0, \ldots, u_i; y_0, \ldots, y_i) \;\propto\; \exp\left[-\frac{1}{2}J_i(x_0; u_0, \ldots, u_i; y_0, \ldots, y_i)\right] \tag{4.2.4}$$

where the symbol $\propto$ stands for "proportional to" and $J_i(x_0; u_0, \ldots, u_i; y_0, \ldots, y_i)$ is given by

$$x_0^*\Pi_0^{-1}x_0 + \sum_{j=0}^{i} u_j^* Q_j^{-1} u_j + \sum_{j=0}^{i}(y_j - H_j x_j)^* R_j^{-1}(y_j - H_j x_j). \tag{4.2.5}$$

In the terminology of [Whi90], any filter that minimizes (4.2.3) is known as a *risk-neutral* filter.

An alternative criterion that is *risk-sensitive* has been extensively studied in [Jac73, SDJ74, Whi90, SFB92] and corresponds to the minimization problem

$$\min_{\{\check{\mathbf{s}}_{j|m}\}} \mu_i(\theta) = \min_{\{\check{\mathbf{s}}_{j|m}\}} \left(-\frac{2}{\theta}\log\left[E\exp(-\frac{\theta}{2}\mathbf{C}_i)\right]\right). \tag{4.2.6}$$

The criterion in (4.2.6) is known as an *exponential cost* criterion, and any filter that minimizes $\mu_i(\theta)$ is referred to as a *risk-sensitive* filter.[2] The scalar parameter $\theta$ is correspondingly called the *risk-sensitivity* parameter. Some intuition concerning the nature of this modified criterion is obtained by expanding $\mu_i(\theta)$ in terms of $\theta$ and writing,

$$\mu_i(\theta) = E(\mathbf{C}_i) - \frac{\theta}{4}Var(\mathbf{C}_i) + O(\theta^2).$$

The above equation shows that for $\theta = 0$, we have the risk-neutral case (*i.e.,* conventional $H^2$ estimation). When $\theta > 0$, we seek to maximize $E\exp(-\frac{\theta}{2}\mathbf{C}_i)$, which is convex and decreasing in $\mathbf{C}_i$. Such a criterion is termed *risk-seeking* (or optimistic) since larger weights are on small values of $\mathbf{C}_i$, and hence we are more concerned with the frequent occurrence of moderate values of $\mathbf{C}_i$ than with the occasional occurrence of large values. When $\theta < 0$, we seek to minimize $E\exp(-\frac{\theta}{2}\mathbf{C}_i)$, which is convex and increasing in $\mathbf{C}_i$. Such a criterion is termed *risk-averse* (or pessimistic) since large weights are on large values of $\mathbf{C}_i$, and hence we are more concerned with the

[2]Such filters are also called LEQG (linear-exponential-quadratic-Gaussian) filters to reflect the facts that, as we shall momentarily see, the optimum filter is linear, the cost is the exponential of a quadratic, and the disturbances are Gaussian.

occasional occurrence of large values than with the frequent occurrence of moderate ones. In what follows, we shall see that in the risk-averse case $\theta < 0$, the limit at which minimizing (4.2.6) makes sense corresponds to the *optimal* $H^\infty$ criterion. [To further compare the risk neutral and risk-averse approaches to estimation we have plotted the cost functions (whose expected values are to be minimized) for these two approaches in Fig. 4.1 (the risk-averse case corresponds to $\theta = 2$). As can be seen in the risk-averse case we incur (exponentially) larger costs for large values of $\mathbf{C}_i$.]

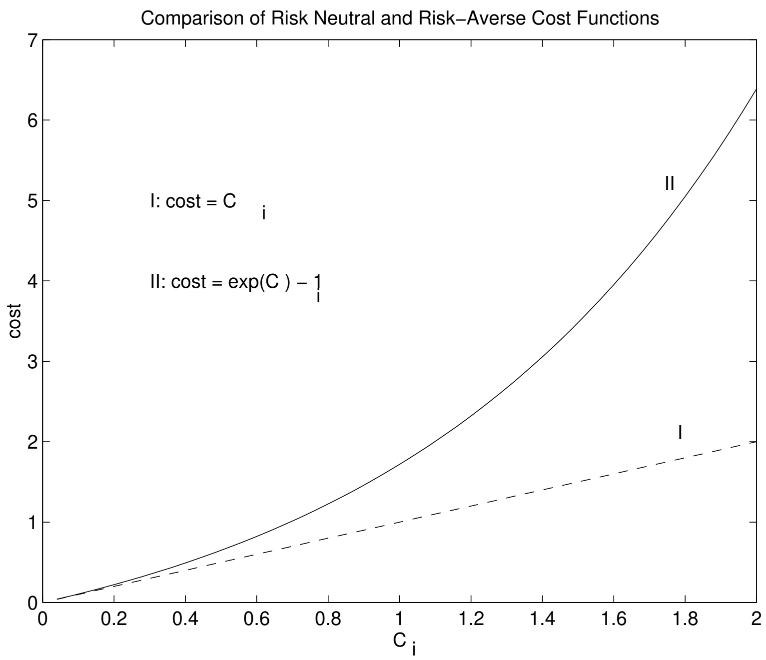


Figure 4.1: Comparison of risk neutral and risk-averse cost functions.

## 4.2.2 Minimizing the Risk-Sensitive Criterion

Using the probability density function (4.2.4) we can easily verify that

$$E\left[\exp(-\frac{\theta}{2}\mathbf{C}_i)\right] \;\propto\; \int \exp(-\frac{\theta}{2}C_i)\exp\left[-\frac{1}{2}J_i(x_0;u_0,\ldots,u_i;y_0,\ldots,y_i)\right] dx_0 \prod_{k=0}^{i} du_k \prod_{k=0}^{i} dy_k,$$

which, upon a change in the order of the integrations, becomes

$$E(\exp(-\frac{\theta}{2}\mathbf{C}_i)) \;\propto\; \int \prod_{k=0}^{i} dy_k \int \exp\left[-\frac{1}{2}\left\{J_i(x_0;u_0,\ldots,u_i;y_0,\ldots,y_i)+\theta C_i\right\}\right] dx_0 \prod_{k=0}^{i} du_k. \tag{4.2.7}$$

Since the $\{\check{z}_{j|m}\}$ are functions of the observations $\{y_k\}_{k=0}^{m}$, the above relation shows that the risk-sensitive criterion (4.2.6) can be alternatively written as

(i) For $\theta > 0$:

$$\max_{\{\check{s}_{j|m}\}} \int \exp\left[-\frac{1}{2}\left\{J_i(x_0;u_0,\ldots,u_i;y_0,\ldots,y_i)+\theta C_i\right\}\right] dx_0 \prod_{k=0}^{i} du_k,$$

(ii) For $\theta < 0$:

$$\min_{\{\check{s}_{j|m}\}} \int \exp\left[-\frac{1}{2}\left\{J_i(x_0;u_0,\ldots,u_i;y_0,\ldots,y_i)+\theta C_i\right\}\right] dx_0 \prod_{k=0}^{i} du_k.$$

This suggests that we define the second-order scalar form

$$\bar{J}_i(x_0,u_0,\ldots,u_i,_0,\ldots,y_i) \triangleq J_i(x_0,u_0,\ldots,u_i,_0,\ldots,y_i)+\theta C_i =$$

$$x_0^*\Pi_0^{-1}x_0+\sum_{j=0}^{i}u_j^*Q_j^{-1}u_j+\sum_{j=0}^{i}(y_j-H_jx_j)^*R_j^{-1}(y_j-H_jx_j)+\theta\sum_{j=0}^{i}(\check{s}_{j|j-1}-L_jx_j)^*(\check{s}_{j|j-1}-L_jx_j)$$

Before proceeding with the extremizations in (i) and (ii), we need to ensure that the integrals in (i) and (ii) be finite. The condition is given by the following Lemma, which is easy to prove.

**Lemma 4.2.1 (Finiteness Condition)** *The integral*

$$\int \exp\left[-\frac{1}{2}\bar{J}_i(x_0;u_0,\ldots,u_i;y_0,\ldots,y_i)\right] dx_0 \prod_{k=0}^{i} du_k,$$

*is finite if, and only if,* $\bar{J}_i(x_0,u_0,\ldots,u_i;y_0,\ldots,y_i)$ *has a minimum over* $\{x_0,u_0,\ldots,u_i\}$. *In this case the value of the integral is proportional to*

$$\exp\left\{-\frac{1}{2}\min_{x_0,u}\bar{J}_i(x_0,u_0,\ldots,u_i;y_0,\ldots,y_i)\right\}.$$

The above Lemma thus reduces the risk-sensitive problem to one of finding the minimum of a scalar quadratic form. More precisely, the criterion becomes

(i) For $\theta > 0$:

$$\min_{\{\check{s}_{j|l}\}} \min_{x_0,u} \bar{J}_i(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i),$$

(ii) For $\theta < 0$:

$$\max_{\{\check{s}_{j|l}\}} \min_{x_0,u} \bar{J}_i(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i).$$

It is noteworthy that the second of the above problems is a quadratic dynamic game (see *e.g.*, [Isa65, BB95]). We shall (very) briefly study quadratic game theory in Sec. 4.3 and shall show that they can also be solved using the Krein space approaches developed in Chapters 2 and 3.

For the time being, however, let us return to the above risk-sensitive problems. In order to solve them, we can introduce the following *auxiliary* Krein state-space model that corresponds to the (possibly indefinite) quadratic form $\bar{J}_i(x_0, u_0, \ldots, u_{i,0}, \ldots, y_i)$.

$$\left\{ \begin{array}{rcl} \mathbf{x}_{j+1} & = & F_j \mathbf{x}_j + G_j \mathbf{u}_j \quad j \geq 0 \\ \begin{bmatrix} \mathbf{y}_i \\ \check{\mathbf{s}}_{i|l} \end{bmatrix} & = & \begin{bmatrix} H_i \\ L_i \end{bmatrix} \mathbf{x}_i + \mathbf{v}_i \end{array} \right. \tag{4.2.8}$$

with

$$\left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_k \\ \mathbf{v}_k \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i \delta_{ij} & 0 \\ 0 & 0 & \begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1} I \end{bmatrix} \delta_{jk} \end{bmatrix}. \tag{4.2.9}$$

We can now readily use the state-space model (4.2.8), and the results of the earlier Chapters 2 and 3, to check for the condition of a minimum over $\{x_0, u_0, \ldots, u_i\}$, and to compute the value at the minimum. Since we have provided the details in those earlier chapters, here we shall only briefly outline the steps for the a posteriori risk-sensitive estimation problem.[3]

[3]Indeed the derivations are almost identical to those of the $H^\infty$ filters of Chapter 3, since the quadratic form $\bar{J}_i(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i)$ is exactly the same as the quadratic forms $J_{f,i}(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i)$, $J_{p,i}(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i)$ and $J_{s,i}(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i)$, when we choose $\theta = \gamma_f^{-2}$, $\theta = \gamma_p^{-2}$ and $\theta = \gamma_s^{-2}$, and when the estimate is chosen as a filtered, predicted and smoothed estimate, respectively.

First note that when $\theta > 0$, the quadratic form $\bar{J}_i(x_0, u_0, \ldots, u_{i,0}, \ldots, y_i)$ is non-negative so that it always has a minimum. Thus risk-seeking optimal filters (corresponding to $\theta > 0$) always exist. On the other hand, when $\theta < 0$, the quadratic form $\bar{J}_i(x_0, u_0, \ldots, u_{i,0}, \ldots, y_i)$ is, in general, indefinite, so that further conditions are required for a minimum to exist. This condition is given by the requirement that the matrices,

$$\begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1}I \end{bmatrix} \quad \text{and} \quad R_{e,j} = \begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1}I \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} \tag{4.2.10}$$

have the same inertia for all $j = 0, \ldots i$, where $P_0 = \Pi_0$, and

$$P_{j+1} = F_j P_j F_j^* + G_j Q_j G_j^* - F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_i F_i^*. \tag{4.2.11}$$

In both cases, the value of $\bar{J}_i(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i)$ at its minimum is given by

$$\bar{J}_i(\min) = \sum_{j=0}^{i} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{bmatrix}, \tag{4.2.12}$$

where $\hat{x}_j$ obeys the Krein space Kalman filter recursion,

$$\hat{x}_{j+i} = F_j\hat{x}_j + K_{p,j} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{bmatrix}, \quad \hat{x}_0 = 0 \tag{4.2.13}$$

with $K_{p,j} = F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1}$. Using a block LDU (lower-diagonal-upper) triangular factorization of the matrix, $R_{e,j}$, and defining the Schur complement,

$$\Delta_j = \theta^{-1}I + L_j P_j L_j^* - L_j P_j H_j^* (R_j + H_j P_j H_j^*)^{-1} H_j P_j L_j^*, \tag{4.2.14}$$

and the estimate,

$$\hat{s}_{j|j} = L_j\hat{x}_j + L_j P_j H_j^* (R_j + H_j P_j H_j^*)^{-1} (y_j - H_j\hat{x}_j), \tag{4.2.15}$$

allows us to write

$$\bar{J}_i(\min) = \sum_{j=0}^{i} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - \hat{s}_{j|j} \end{bmatrix}^* \begin{bmatrix} R_j^{-1} & 0 \\ 0 & \Delta_j^{-1} \end{bmatrix} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{bmatrix}. \tag{4.2.16}$$

Thus in the risk-seeking case ($\theta > 0$), our problem becomes,

$$min_{\{\check{s}_{j|j}\}} \sum_{j=0}^{i} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - \hat{s}_{j|j} \end{bmatrix}^* \begin{bmatrix} R_j^{-1} & 0 \\ 0 & \Delta_j^{-1} \end{bmatrix} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{bmatrix}, \tag{4.2.17}$$

and in the risk-averse case ($\theta < 0$),

$$max_{\{\check{s}_{j|j}\}} \sum_{j=0}^{i} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - \hat{s}_{j|j} \end{bmatrix}^* \begin{bmatrix} R_j^{-1} & 0 \\ 0 & \Delta_j^{-1} \end{bmatrix} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{bmatrix}. \tag{4.2.18}$$

But when $\theta > 0$, we have $R_{e,j} > 0$, and thus $\Delta_j > 0$. Therefore the quadratic form in (4.2.17) has a minimum over the $\{\check{s}_{j|j}\}$ which is readily given by

$$\check{s}_{j|j} = \hat{s}_{j|j}. \tag{4.2.19}$$

Likewise, when $\theta < 0$, due to the inertia condition on $R_{e,j}$, we have $\Delta_j < 0$. Thus, in this case, the quadratic form in (4.2.18) has a maximum over the $\{\check{s}_{j|j}\}$ which is also given by

$$\check{s}_{j|j} = \hat{s}_{j|j}. \tag{4.2.20}$$

As in Chapter 3 it is also possible to give a recursion for the $\hat{x}_{j|j}$ which yields $L_j\hat{x}_{j|j} = \hat{s}_{j|j}$. This leads to the following result.

**Theorem 4.2.1 (A Posteriori Risk-Sensitive Filter)** *Consider the a posteriori risk-sensitive estimation problem,*

$$\min_{\{\check{\mathbf{s}}_{j|j}\}} \; -\frac{2}{\theta}\log\left[E\exp\left(-\frac{\theta}{2}\sum_{j=0}^{i}(\mathbf{s}_j - \hat{\mathbf{s}}_{j|j})^*(\mathbf{s}_j - \hat{\mathbf{s}}_{j|j})\right)\right],$$

*for some given $\theta$. When $\theta > 0$, the risk-sensitive estimation problem always has a solution. When $\theta < 0$, a solution exists if, and only if, the matrices*

$$\begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1}I \end{bmatrix} \quad \textit{and} \quad R_{e,j} = \begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1}I \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix}$$

*have the same inertia for all $j = 0, 1, \ldots i$, where $P_0 = \Pi_0$ and*

$$P_{j+1} = F_jP_jF_j^* + G_jQ_jG_j^* - F_jP_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_iF_i^*.$$

*In both cases the optimal risk-sensitive filter with parameter $\theta$ is given by*

$$\check{s}_{j|j} = L_j \hat{x}_{j|j},$$

*where $\hat{x}_{j|j}$ satisfies the recursion,*

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + K_{s,j+1}(y_{j+1} - H_{j+1} F_j \hat{x}_{j|j}) \quad , \quad \hat{x}_{-1|-1} = 0$$

*and*

$$K_{s,j+1} = P_{j+1} H_{j+1}^* (I + H_{j+1} P_{j+1} H_{j+1}^*)^{-1}.$$

Similar results hold for the a priori, smoothed, and $l$-step ahead, estimation problems, as given below. The proofs are similar and are omitted for brevity.

**Theorem 4.2.2 (A Priori Risk-Sensitive Filter)** *Consider the a priori risk-sensitive estimation problem,*

$$\min_{\{\check{s}_j\}} \; -\frac{2}{\theta}\log\left[E\exp\left(-\frac{\theta}{2}\sum_{j=0}^{i}(\mathbf{s}_j - \hat{\mathbf{s}}_j)^*(\mathbf{s}_j - \hat{\mathbf{s}}_j)\right)\right],$$

*for some given $\theta$. When $\theta > 0$, the a priori risk-sensitive estimation problem always has a solution. When $\theta < 0$, a solution exists if, and only if, all leading submatrices of*

$$\begin{bmatrix} \theta^{-1}I & 0 \\ 0 & R_j \end{bmatrix} \quad \textit{and} \quad R_{e,j} = \begin{bmatrix} \theta^{-1}I & 0 \\ 0 & R_j \end{bmatrix} + \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix}$$

*have the same inertia for all $j = 0, 1, \ldots i$, where $P_j$ is the same as in the a posteriori problem. In both cases the a priori risk-sensitive filter with parameter $\theta$ is given by*

$$\check{s}_{j|j-1} = L_j \hat{x}_{j|j-1}$$

*where $\hat{x}_j$ satisfies the recursion,*

$$\hat{x}_{j+1|j} = F_j \hat{x}_{j|j-1} + K_{a,j}(y_j - H_j \hat{x}_{j|j-1}) \quad , \quad \hat{x}_{0|-1} = 0$$

*with*

$$K_{a,j} = F_j \tilde{P}_j H_j^* (R_j + H_j \tilde{P}_j H_j^*)^{-1}, \quad \textit{and} \quad \tilde{P}_j = P_j - P_j L_j^* (\theta^{-1} I + L_j P_j L_j^*)^{-1} L_j P_j.$$

**Theorem 4.2.3 (Risk-Sensitive Smoother)** *Consider the smoothed risk-sensitive estimation problem,*

$$\min_{\{\check{\mathbf{s}}_{j|i}\}} \; -\frac{2}{\theta}\log\left[E\exp\left(-\frac{\theta}{2}\sum_{j=0}^{i}(\mathbf{s}_j-\hat{\mathbf{s}}_{j|i})^*(\mathbf{s}_j-\hat{\mathbf{s}}_{j|i})\right)\right],$$

*for some given $\theta$. When $\theta > 0$, the smoothed risk-sensitive problem always has a solution. When $\theta < 0$, a solution exists if, and only if, the block diagonal matrix*

$$R_e = R_{e,0} \oplus R_{e,1} \oplus \ldots \oplus R_{e,i}$$

*where*

$$R_{e,j} = \begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1}I \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix}$$

*and $P_j$ is the same as in the a posteriori problem, has $(i+1)p$ positive eigenvalues and $(i+1)q$ negative eigenvalues. In other words, if, and only if,*

$$In[R_e] = \begin{bmatrix} (i+1)p & 0 & (i+1)q \end{bmatrix}.$$

*In both cases the risk-sensitive smoother is the same as the $H^2$ smoother.*

**Theorem 4.2.4 (Risk-Sensitive $l$-Step Predictor)** *Consider the $l$-step ahead risk-sensitive estimation problem,*

$$\min_{\{\check{\mathbf{s}}_{j|j-l}\}} \; -\frac{2}{\theta}\log\left[E\exp\left(-\frac{\theta}{2}\sum_{j=0}^{i}(\mathbf{s}_j-\hat{\mathbf{s}}_{j|j-l})^*(\mathbf{s}_j-\hat{\mathbf{s}}_{j|j-l})\right)\right],$$

*for some given $\theta$ and $l > 0$. When $\theta > 0$, the $l$-step ahead risk-sensitive problem always has a solution. When $\theta < 0$, a solution exists if, and only if, for each $j = 0, \ldots, i-l$,*

*(i) the two matrices*

$$R_{e,j} = \begin{bmatrix} I_p & 0 \\ 0 & \theta^{-1}I_q \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} I_p & 0 \\ 0 & \theta^{-1}I_q \end{bmatrix}$$

*have the same inertia, where $P_j$ is the same as in the a posteriori problem.*

*(ii) the sequence of matrices*

$$R_{e,j+m}^{j+1} \triangleq \theta^{-1} I_q + L_{j+m} P_{j+m}^{j+1} L_{j+m}^*, \quad m = 1, \dots l$$

*are negative definite, where* $P_{j+m}^{j+1}$ *satisfies the Riccati recursion*

$$\begin{array}{rcl} P_{j+m+1}^{j+1} & = & F_{j+m} P_{j+m}^{j+1} F_{j+m}^* + G_{j+m} Q_{j+m} G_{j+m}^* - \\ & & F_{j+m} P_{j+m}^{j+1} L_{j+m}^* \left[ R_{e,j+m}^{j+1} \right]^{-1} L_{j+m} P_{j+m}^{j+1} F_{j+m}^* \end{array}, \quad P_{j+1}^{j+1} = P_{j+1}.$$

*In either case, the risk-sensitive l-step predictor is given by*[4]

$$\check{s}_{j+l|j} = L_{j+l} \hat{x}_{j+l}^{j+1},$$

*where* $\hat{x}_{j+m}^{j+1}$ *satisfies the recursion,*

$$\hat{x}_{j+m+1}^{j+1} = F_{j+m} \hat{x}_{j+m}^{j+1} + K_{p,j+m}^{j+1} (\check{s}_{j+m|j+m-l} - L_{j+m} \hat{x}_{j+m}^{j+1}), \quad \hat{x}_{j+1}^{j+1} = \hat{x}_{j+1}$$

*with*

$$K_{p,j+m}^{j+1} = F_{j+m} P_{j+m}^{j+1} L_{j+m}^* \left[ R_{e,j+m}^{j+1} \right]^{-1},$$

*and where* $\hat{x}_j$ *satisfies the recursion,*

$$\hat{x}_{j+1} = F_j \hat{x}_j + K_{p,j} \left[ \begin{array}{c} y_j - H_j \hat{x}_j \\ \check{s}_{j|j-l} - L_j \hat{x}_j \end{array} \right], \quad \hat{x}_0 = \check{x}_0$$

*with*

$$K_{p,j} = F_j P_j \left[ \begin{array}{cc} H_j^* & L_j^* \end{array} \right] R_{e,j}^{-1}.$$

We can now state the striking resemblances between the $H^\infty$ and the risk-sensitive filters. The *central* $H^\infty$ filters obtained earlier are essentially risk-sensitive filters with parameter $\theta = -\gamma^{-2}$.[5] Note, however, that at each level $\gamma$, the $H^\infty$ filters are not

[4] We also have $\check{s}_{k|k-l} = L_k F_{k-1} \dots F_0 \check{x}_0$ for $k = 0, \dots, l-1$.

[5] This leads to a stochastic interpretation of the central $H^\infty$ filters (and controllers). In contrast to $H^2$ filters (and controllers) that minimize an expected quadratic cost, the central $H^\infty$ filters (and controllers) minimize an expected exponential-quadratic cost. We shall have more to say about this interpretation in Chapter 9. Here instead let us recall that we had earlier, *i.e.*, in Chapter 1, obtained another stochastic interpretation of the central $H^\infty$ filters and controllers — namely that they are maximum entropy solutions.

unique, whereas for each $\theta$, the risk-sensitive filters are unique. Also, the risk-sensitive filters generalize to the $\theta > 0$ case. It is also noteworthy that the *optimal* $H_\infty$ filter corresponds to the risk-sensitive filter with $\bar{\theta} = -\gamma_{opt}^{-2}$, and that $\bar{\theta}$ is that value for which the minimizing property of $\bar{J}_i$ breaks down and $\mu_i(\theta)$ becomes infinite. This relationship between the optimal $H_\infty$ filter and the corresponding risk-sensitive filter was first noted in [GD88].

### 4.2.3 Risk-Sensitive Control

We should also mention that the risk-sensitive framework can be used to study control problems. To this end, consider the (possibly time-varying) state-space model,

$$\begin{cases} \mathbf{x}_{i+1} = F_i\mathbf{x}_i + G_{1,i}\mathbf{w}_i + G_{2,i}\mathbf{u}_i \\ \mathbf{s}_i = L_i\mathbf{x}_i \end{cases}, 0 \leq i \leq N \tag{4.2.21}$$

where $\{\mathbf{w}_i\}$ is the exogenous input, which is assumed to be a zero-mean Gaussian stochastic process with variance,

$$E\mathbf{w}_i\mathbf{w}_j^* = Q_i\delta_{ij}, \tag{4.2.22}$$

$\mathbf{s}_i$ is the signal we intend to regulate, and $\mathbf{u}_i$ is the control signal used to influence the dynamics of the system. Moreover, the initial state, $\mathbf{x}_0$, is also a zero-mean Gaussian random variable, independent of the $\{\mathbf{w}_i\}$, with variance, $\Pi_0$.

Recall that in the $H^2$ case, the control signals, $\{\mathbf{u}_i\}$ were chosen so as to minimize the expected value of the quadratic cost,

$$\mathbf{J}_N^c = \sum_{j=0}^{N} \mathbf{s}_i^* R_i^c \mathbf{s}_i + \sum_{j=0}^{N} \mathbf{u}_i^* Q_i^c \mathbf{u}_i + \mathbf{x}_{N+1} P_{N+1}^c \mathbf{x}_{N+1}, \tag{4.2.23}$$

where the $\{Q_i^c, R_i^c\}$ and $P_{N+1}^c$ are given positive semidefinite weighting matrices.

In risk-sensitive control, as in risk-sensitive estimation, the objective is to choose the controls so as to minimize the expected value of an exponential-quadratic cost, *i.e.*,

$$\min_{\{\mathbf{u}_i\}} \left( -\frac{2}{\theta}\log\left[E\exp(-\frac{\theta}{2}\mathbf{J}_N^c)\right]\right), \tag{4.2.24}$$

where $\theta$ is, once more, referred to as the risk-sensitivity parameter. The remarks presented at the end of Sec. 4.2.1 on risk-neutral, risk-seeking, and risk-averse estimators apply to the controllers that satisfy (4.2.24) in the exact same way, and therefore will not be repeated here.

Moreover, we should stress that, as was the case in Sec. 1.5, we can have either full information or measurement feedback risk-sensitive controllers. When the control signals that satisfy (4.2.24) are allowed to be causal functions of the states and exogenous inputs, *i.e.*, the $\{\mathbf{x}_i, \mathbf{w}_i\}$, then the controller is referred to as a full information controller. In the measurement feedback case, $\mathbf{u}_i$ is allowed only to be a causal function of a certain observation process,

$$\mathbf{y}_i = H_i \mathbf{x}_i + \mathbf{v}_i, \tag{4.2.25}$$

where $\{\mathbf{v}_i\}$ is a zero-mean Gaussian random process such that

$$E \begin{bmatrix} \mathbf{w}_i \\ \mathbf{v}_i \\ \mathbf{x}_0 \end{bmatrix} \begin{bmatrix} \mathbf{w}_j^* & \mathbf{v}_j^* & \mathbf{x}_0^* \end{bmatrix} = \begin{bmatrix} Q_i \delta_{ij} & 0 & 0 \\ 0 & R_i \delta_{ij} & 0 \\ 0 & 0 & \Pi_0 \end{bmatrix}. \tag{4.2.26}$$

We shall not go any further into the details of risk-sensitive control here. We just mention, in passing, that its solution is also amenable to the Krein space techniques studied so far. Control problems (especially of the $H^\infty$ type) will be studied in Chapter 6.

## 4.3 Quadratic Dynamic Games

Game theory is a vast subject with numerous applications in economics and the social sciences [NM44, KT50, Mor94, Col95]. More recently, there has also been considerable use of game-theoretic ideas and methods in estimation and control [Isa65, Ber64, BH69, KS88, Pon90, BO82]. Even a brief glimpse at this field is well beyond the intentions and scope of this thesis. However, a small subset of game-theoretical problems, namely the class of quadratic dynamic games, is closely related to the problems studied in this thesis and lends itself to analysis by the methods that we have used so far.[6] The reason for this is that quadratic games can be immediately related to certain indefinite quadratic forms whose properties can be studied via Krein space methods. Indeed the (min-max) solutions to such games are found by stationarizing these quadratic forms and checking for certain minimizing and maximizing conditions.

### 4.3.1 General Remarks

To gain an understanding of quadratic games, consider the quadratic cost function,

$$J(a,b) = \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{4.3.1}$$

where $a \in \mathcal{C}^n$ and $b \in \mathcal{C}^m$ are arbitrary vectors and $\{A, B, C\}$ are given matrices (with $A$ and $C$ Hermitian). Suppose now that there are two players, say player $I$ and player $II$, where player $I$ can choose $a$ and would like to do so to minimize $J(a,b)$, and where player $II$ would like to maximize $J(a,b)$ through its choice of $b$. The central question here (and of game theory in general) is what the optimal strategies of the players should be.

To this end, suppose that player $I$ has access to player $II$'s choice of $b$. Therefore player $I$ will choose $a$ such that $J(a,b)$ is minimized for that value of $b$. In this case, player $II$'s best strategy will be to choose $b$ such that this minimum value of $J(a,b)$ is maximized. This then leads to the following (so-called max-min) problem,

$$\max_b \ \min_a \ \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{4.3.2}$$

Now clearly the condition for having a (unique) minimum over $a$ is that $A > 0$. Once this is the case it easy to see that

$$\min_a \ J(a,b) = b^*(C - B^*A^{-1}B)b \quad \text{and} \quad \arg\min_a \ J(a,b) = A^{-1}Bb. \tag{4.3.3}$$

Now $b^*(C - B^*A^{-1}B)b$ will have a maximum over $b$ if, and only if, $C - B^*A^{-1}B < 0$.

[6] Incidentally, quadratic dynamic games are the ones that have been most studied in control theory, and have turned out to be closely related to $H^\infty$ and risk-sensitive control.

Thus, we conclude that

$$\max_b \min_a \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0, \tag{4.3.4}$$

and that the optimal max-min strategy is $a = 0$ and $b = 0$. Moreover the condition for the existence of a max-min solution is that

$$A > 0 \quad \text{and} \quad C - B^*A^{-1}B < 0. \tag{4.3.5}$$

Of course, we can also reverse the situation (and assume that player $II$ has access to player $I$'s choice) so that we are led to the (so-called min-max) problem,

$$\min_a \max_b \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{4.3.6}$$

Using a similar argument, we can see that

$$\min_a \max_b \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0, \tag{4.3.7}$$

and that the optimal min-max strategy is $a = 0$ and $b = 0$. The condition for the existence of a min-max strategy, however, now is

$$C < 0 \quad \text{and} \quad A - BC^{-1}B^* > 0. \tag{4.3.8}$$

Note that although the max-min and min-max strategies have turned out to be the same (which is generally the case, since they are both found from stationarizing the quadratic form, $J(a, b)$), the existence conditions for the two solutions are different (since one problem requires first a minimum over $a$ and then a maximum over $b$, and the other problem the vice versa). The condition that the max-min and min-max solutions exist simultaneously is called the *saddle point* condition, which using (4.3.5) and (4.3.8) is readily seen to be

$$A > 0 \quad \text{and} \quad C < 0. \tag{4.3.9}$$

[The above two conditions are obviously necessary for (4.3.5) and (4.3.8) to hold. They are also sufficient, since they imply $C - B^*A^{-1}B < 0$ and $A - BC^{-1}B^* > 0$.]

In this case the optimal strategy, say $(a_{opt}, b_{opt})$, is called the saddle point strategy, and has the property that,

$$J(a, b_{opt}) \geq J(a_{opt}, b_{opt}) \geq J(a_{opt}, b), \tag{4.3.10}$$

for all possible strategies, $(a, b)$. Indeed it is straightforward to see,

$$J(a, b_{opt}) = J(a, 0) = a^* A a \geq 0 = J(a_{opt}, b_{opt}) = 0 \geq b^* C b \geq J(0, b) = J(a_{opt}, b).$$

In other words, either player may lose by choosing a strategy differing from the saddlepoint strategy.

The above problem is, of course, very simplistic[7]. However, it serves to illustrate the major points of quadratic game theory. Before listing these major points, it will be useful to consider a slight generalization of the above problem, in which the cost function is replaced by

$$J(a, b, y) = \begin{bmatrix} a^* & b^* & y^* \end{bmatrix} \begin{bmatrix} A & B & D \\ B^* & C & E \\ D^* & E^* & F \end{bmatrix} \begin{bmatrix} a \\ b \\ y \end{bmatrix}, \tag{4.3.11}$$

in which $y \in \mathcal{C}^p$ is a vector which players $I$ and $II$ both have access to, and $\{D, E, F\}$ are given matrices (with $F$ Hermitian). In this case, it is straightforward to see that the stationary point of $J(a, b, y)$ over $(a, b)$ is given by

$$\begin{bmatrix} a_{opt} \\ b_{opt} \end{bmatrix} = - \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}^{-1} \begin{bmatrix} D \\ E \end{bmatrix} y, \tag{4.3.12}$$

and that the value of $J(a, b, y)$ at the stationary point is,

$$J(a_{opt}, b_{opt}, y) = y^* \left\{ F - \begin{bmatrix} D^* & E^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}^{-1} \begin{bmatrix} D \\ E \end{bmatrix} \right\} y. \tag{4.3.13}$$

The condition for this stationary point to be a max-min, min-max, or saddle point solution is still given by (4.3.5), (4.3.8) or (4.3.9), respectively.

Using the above examples, we can now list the major points of quadratic game theory as follows:

---

[7]It is called a *static* quadratic game.

(i) The cost function is an *indefinite* quadratic form.

(ii) The solution to the game (be it a max-min, min-max, or saddle point solution) is found by stationarizing the indefinite quadratic form over the strategies of the two players (since max-min points, min-max points, and saddle points are all special cases of a stationary point).

(iii) The condition for the existence of a solution of a certain kind (such as a max-min, min-max, or saddle point solution) is given by a certain inertia condition (such as (4.3.5), (4.3.8) or (4.3.9)) on the coefficient matrix of the game's indefinite quadratic form.

The above remarks indicate the connections between quadratic game theory and the Krein space approach developed in this thesis. To solve a quadratic game, using this approach, all we need to do is identify a Krein space model with the indefinite quadratic form of the game, and to then compute the stationary point via a Krein space projection. The conditions for the existence of a max-min, min-max, or saddle point solution is then given by various inertia conditions on the Gramians of certain Krein space variables. [For example, if the problem has state-space structure, the projection (hence the stationary point) can be computed via the Krein space Kalman filter, and the inertia conditions can be checked from variables that are by-products of the Kalman recursions (see Sec. 2.7).]

To illustrate the above remarks, consider the cost function, $J(a,b,y)$, of (4.3.11). To identify a Krein space model with it, we can use the approach of Sec. 2.5. To this end, let us define $s \triangleq \begin{bmatrix} a \\ b \end{bmatrix}$ and the Krein space variables, $\{\mathbf{s},\mathbf{y}\}$, such that

$$\left\langle \begin{bmatrix} \mathbf{s} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{s} \\ \mathbf{y} \end{bmatrix} \right\rangle = \begin{bmatrix} R_s & R_{sy} \\ R_{ys} & R_y \end{bmatrix} \triangleq \begin{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} & \begin{bmatrix} D \\ E \end{bmatrix} \\ \begin{bmatrix} D^* & E^* \end{bmatrix} & F \end{bmatrix}^{-1}. \tag{4.3.14}$$

This then allows us to write,

$$J(a,b,y) = J(s,y) = \begin{bmatrix} s^* & y^* \end{bmatrix} \begin{bmatrix} R_s & R_{sy} \\ R_{ys} & R_y \end{bmatrix}^{-1} \begin{bmatrix} s \\ y \end{bmatrix}. \tag{4.3.15}$$

Using Theorem 2.5.2, the stationary point of $J(s,y)$ over $s$, which we denote by $\hat{s}$, is given by the Krein space projection of $\mathbf{s}$ on $\mathcal{L}(\mathbf{y})$, *i.e.*,

$$\hat{s} = R_{sy} R_y^{-1} y. \tag{4.3.16}$$

Moreover, the conditions for the above stationary point to be a max-min, min-max, or saddlepoint solution are related to the inertia of the matrix,

$$(R_s - R_{sy} R_y^{-1} R_{ys})^{-1} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}. \tag{4.3.17}$$

[Recall that methods to determine this inertia have been developed in Secs. 2.5.3 and 2.6.1.]

We close this section with one more remark. The quadratic games considered so far were all static. When the vectors $a$, $b$ and $y$ are aggregate vectors of some time series, $\{a_i\}$, $\{b_i\}$ and $\{y_i\}$, *i.e.*,

$$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}, \quad b = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_N \end{bmatrix}, \quad y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix} \tag{4.3.18}$$

that are dynamically related (say, via a state-space model), then the game is referred to as a quadratic *dynamic* game. In such games, both players (or occasionally one of the two) have only causal access to the observations, $\{y_i\}$, or to the other player's strategy. In other words, the strategy chosen at time $i$ by, say, player $I$, (which we recall is just $a_i$) is only allowed to be a function of current and past observations, $\{y_j, j \leq i\}$ (and occasionally of current and past strategies of player $II$, $\{b_j, j \leq i\}$). This causality restriction affects both the structure of the optimal strategies and the actual conditions for the existence of a max-min, min-max or saddle point solution. Although the solutions in this case are somewhat more complex, they can either be obtained via a canonical factorization approach[8] of the matrix, $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$, (along

[8]The relationship between game theory and indefinite factorizations were apparently first studied by Yakubovich [Yak70b] and Banker [Ban73].

the lines of Secs. 1.3.1 and 1.6.2) or by performing a recursive stationarization of the quadratic form $J(a, b, y)$ (along the lines of Sec. 2.7). In summary, there is no essential difference between the methods for solving dynamic games with a causality restriction and the methods for finding causal estimators and controllers that have been studied so far.

### 4.3.2 Specific Examples

We now very briefly turn to some special cases of the general quadratic games described above. The examples we consider are in the contexts of (state-space) estimation and control and are intimately related to the $H^\infty$ and risk-sensitive formulations studied earlier. In essence, we shall see that the central $H^\infty$ filters and controllers, and the risk-sensitive filters and controllers, can be considered as the solutions to certain quadratic dynamic games.

#### Estimation Problems

Consider the standard state-space model

$$\left\{\begin{array}{rcl} x_{i+1} & = & F_i x_i + G_i u_i, \quad x_0 \\ y_i & = & H_i x_i + v_i \\ s_i & = & L_i x_i \end{array}\right. \qquad 0 \leq i \leq N \tag{4.3.19}$$

where, as usual, the initial condition $x_0$, and the disturbances, $\{u_i, v_i\}$, are unknown, $\{y_i\}$ is the observations sequence, and $\{s_i\}$ is the signal we intend to estimate (using the observations). In the game-theoretic approach to estimation, the unknown disturbances $\{x_0, \{u_, v_i\}_{i=0}^N\}$ are considered to be an adversary whose objective is to disrupt our attempt at estimating the $\{s_i\}$. To be more specific, consider the a posteriori filtering problem where the estimates are given by $\check{s}_{i|i} = \mathcal{F}_{f,i}(y_0, \ldots, y_i)$, with $\mathcal{F}_f(\cdot)$ being the estimation strategy that must be determined. Now one measure of how well our estimation scheme is performing is the estimation error energy,

$$\sum_{i=0}^N (s_i - \check{s}_{i|i})^* (s_i - \check{s}_{i|i}). \tag{4.3.20}$$

Roughly speaking, our objective is to choose the estimator, $\mathcal{F}_f(\cdot)$ so as to minimize the estimation error energy, whereas nature's (our adversary's) objective is to choose the disturbances so as to maximize it. To put the situation on an equal footing, one must include a penalty on the choice of large disturbances (otherwise nature could make the estimation error arbitrarily large by choosing arbitrarily large disturbances). This is achieved by adding a negative definite quadratic term to the above cost that penalizes large disturbances. Thus we are led to the following cost function,

$$J_N = \sum_{i=0}^{N}(s_i - \check{s}_{i|i})^*(s_i - \check{s}_{i|i}) - \gamma^2\left(x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i\right), \quad (4.3.21)$$

where $\Pi_0$, $Q_i$ and $R_i$ are given positive definite weighting matrices, and where $\gamma$ is a scalar that determines the respective contributions of the estimation error and disturbance energies to the cost function $J_N$.

The game-theoretic estimation problem can be thus formulated as follows,

$$\min_{\{\check{s}_{i|i}\}} \max_{x_0,\{u_i\},\{v_i\}} \sum_{i=0}^{N}(s_i - \check{s}_{i|i})^*(s_i - \check{s}_{i|i}) - \gamma^2\left(x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i\right), \quad (4.3.22)$$

where $\check{s}_{i|i} = \mathcal{F}_{f,i}(y_0, \ldots, y_i)$ and the $\{x_0, \{u_i\}, \{v_i\}, \{y_i\}, \{s_i\}\}$ are related via the state-space model (4.3.19).

Now the quadratic form, $J_N$, is an indefinite quadratic form of the standard type that has been studied since Chapter 2. Therefore we can readily use the Krein space Kalman filter to compute the stationary point of $J_N$ and to check whether the stationary point has the desired (maximum over $\{x_0, \{u_i\}, \{v_i\}\}$, minimum over $\{\check{s}_{i|i}\}$) properties (via certain inertia conditions obtained from the Kalman filter recursions). Indeed further inspection of $J_N$ reveals that, apart from a minus sign, it is the exact same indefinite quadratic form that was studied in the $H^\infty$ and risk-sensitive a posteriori estimation problems. Following through with the arguments presented there, (which we will not repeat here) shows that the central $H^\infty$ and risk-sensitive a posteriori filters are, in fact, given by the solution to the game (4.3.22).[9] Similar statements hold for the a priori, smoothed, and $l$-step ahead central $H^\infty$ and risk-sensitive filters as well.

[9]This observation is the whole premise of the game-theoretic approach to $H^\infty$ estimation and control (see *e.g.*, [BB95, LAKG92]).

The above remarks conclude our brief study of game-theoretic estimation and we now turn our attention to control.

**Control Problems**

Consider the state-space model,

$$\begin{cases} x_{i+1} = F_i x_i + G_{1,i} w_i + G_{2,i} u_i, \quad x_0 \\ s_i = L_i x_i \end{cases} \qquad 0 \le i \le N \tag{4.3.23}$$

where $\{w_i\}$ is the exogenous input, $\{s_i\}$ is the signal we want to regulate, and $\{u_i\}$ is the control input. In the full information problem, the control signal $u_i$ is allowed to be a function of the initial state, $x_0$, and of the current and past exogenous inputs, $\{w_j, j \le i\}$. In the measurement feedback problem, all we have access to is the measurement signal,

$$y_i = H_i x_i + v_i, \tag{4.3.24}$$

where $\{v_i\}$ is an (unknown) measurement disturbance, and therefore the control signal $u_i$ is only allowed to be a function of current and past observations, $\{y_j, j \le i\}$. In either case, the control objective is to minimize the quadratic cost,

$$\sum_{j=0}^{N} s_i^* R_i^c s_i + \sum_{j=0}^{N} u_i^* Q_i^c u_i + x_{N+1} P_{N+1}^c x_{N+1}. \tag{4.3.25}$$

In the game-theoretic approach to control the disturbances $x_0$, $\{w_i\}$, and in the case of measurement feedback control, $\{v_i\}$, are considered to be adversaries that choose to maximize the above cost. As in estimation, to level the playing field, we must penalize the choice of large values of disturbances on the part of nature (our opponent). This is achieved by adding a negative definite quadratic term to the above cost that penalizes large disturbances. We are thus led to the following cost functions: for the full information problem,

$$J_N^c = \sum_{j=0}^{N} s_i^* R_i^c s_i + \sum_{j=0}^{N} u_i^* Q_i^c u_i + x_{N+1} P_{N+1}^c x_{N+1} - \gamma^2 \left( x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \right), \tag{4.3.26}$$

and for the measurement feedback problem,

$$J_N^c = \sum_{j=0}^{N} s_i^* R_i^c s_i + \sum_{j=0}^{N} u_i^* Q_i^c u_i + x_{N+1} P_{N+1}^c x_{N+1} - \gamma^2 \left( x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i \right), \tag{4.3.27}$$

where $\Pi_0$, $Q_i$ and $R_i$ are given positive definite weighting matrices, and where $\gamma$ is a scalar that determines the respective contributions of the original quadratic cost and disturbance energies to the cost function $J_N^c$.

The game-theoretic estimation problem can be thus formulated as follows: for the full information problem,

$$\min_{\{u_i\}} \max_{x_0, \{u_i\}} J_N^c, \tag{4.3.28}$$

where $u_i = \mathcal{F}_i(x_0, w_0, \ldots, w_i)$, $J_N^c$ is given by (4.3.26), and the $\{s_i\}$, and $\{x_0, \{w_i, u_i\}\}$ are related via (4.3.23); for the measurement feedback problem,

$$\min_{\{u_i\}} \max_{x_0, \{u_i\}, \{v_i\}} J_N^c, \tag{4.3.29}$$

where $u_i = \mathcal{F}_i(y_0, \ldots, y_i)$, $J_N^c$ is given by (4.3.27), and the $\{s_i, y_i\}$ and $\{x_0, \{w_i, u_i, v_i\}\}$ are related via (4.3.23) and (4.3.24).

The quadratic form, $J_N^c$, is an indefinite quadratic form, but not of the type that has been studied since Chapter 2. In Chapter 6 we shall show that this quadratic form can be motivated, and introduced, via the concept of a dual basis, and that its stationary point can be obtained by certain projections, not in the original spaces, but in the resulting dual spaces. [This is yet another manifestation of the duality between estimation and (full information) control, that we first noted in Sec. 1.5.1. In fact, the aforementioned duality is geometric while the one presented in Sec. 1.5.1 was purely algebraic.] We shall not give the details of how to compute these stationary points (or how to verify the min-max or saddle point conditions) here, since that topic will be studied in length in Chapter 6. Instead, we close this section by remarking that the indefinite quadratic form $J_N^c$ is exactly the same (apart from a minus sign) as the indefinite quadratic form that is obtained from $H^\infty$ and risk-sensitive control. In particular, it turns out that the full information and measurement feedback $H^\infty$ and risk-sensitive controllers coincide with the solutions to the quadratic dynamic games (4.3.28) and (4.3.29), respectively.

## 4.4 Finite Memory Adaptive Filtering

We now consider an application of the Krein space Kalman filter to the problem of finite memory (or sliding window) adaptive filtering.[10] It has been recently shown [SK94b] that a unified derivation of adaptive filtering algorithms, and their corresponding fast versions, can be obtained by properly recasting the adaptive problem into a standard state-space estimation problem. We now verify that if we further allow the elements of the state-space model to belong to a Krein space, then the so-called sliding window or finite-memory problem can also be handled within the same framework. In fact, this framework also allows us to easily consider more general sliding patterns with windows of varying lengths, as explained ahead. Moreover, we shall obtain a physical interpretation of innovations with negative Gramian, and see that it corresponds to the loss of information.

### 4.4.1 The Standard Problem

The finite memory adaptive filtering problem can be formulated as follows: given the input-output pairs $\{h_j, d_j\}$ where $h_j \in \mathcal{C}^{1\times n}$ is a known input vector and $d_j \in \mathcal{C}$ is a known output scalar, recursively determine estimates of an unknown weight vector $w \in \mathcal{C}^n$, such that the scalar quadratic form

$$J_i(w, d_{i-l_i+1}, \ldots, d_i, h_{i-l_i+1}, \ldots, h_i) = w^*\Pi_0^{-1}w + \sum_{j=i-l_i+i}^{i}(d_j - h_j w)^*(d_j - h_j w), \quad (4.4.1)$$

with $\Pi_0 > 0$ a given weighting matrix and the $\{l_i\}$ given (time-varying) window lengths, is minimized for each $i$.

Since $J_i$ is a function of the pairs $\{h_j, d_j\}_{j=i-l_i+i}^{i}$, at each time instant $i$, we are interested in determining the estimate of $w$ using only the data given over an interval of length $l_i$. The quantity $l_i \geq 0$ is therefore referred to as the length (or memory) of the sliding window.

Note that we allow for a time-variant window length. To clarify this point, consider the example of Fig. 4.2 where at time $i$ we have a window of length $l_i = l$. At the next

[10]Refer back to Sec. 1.2.1 for a brief introduction to adaptive filtering. The topic will be further studied in Chapters 9 and 11.

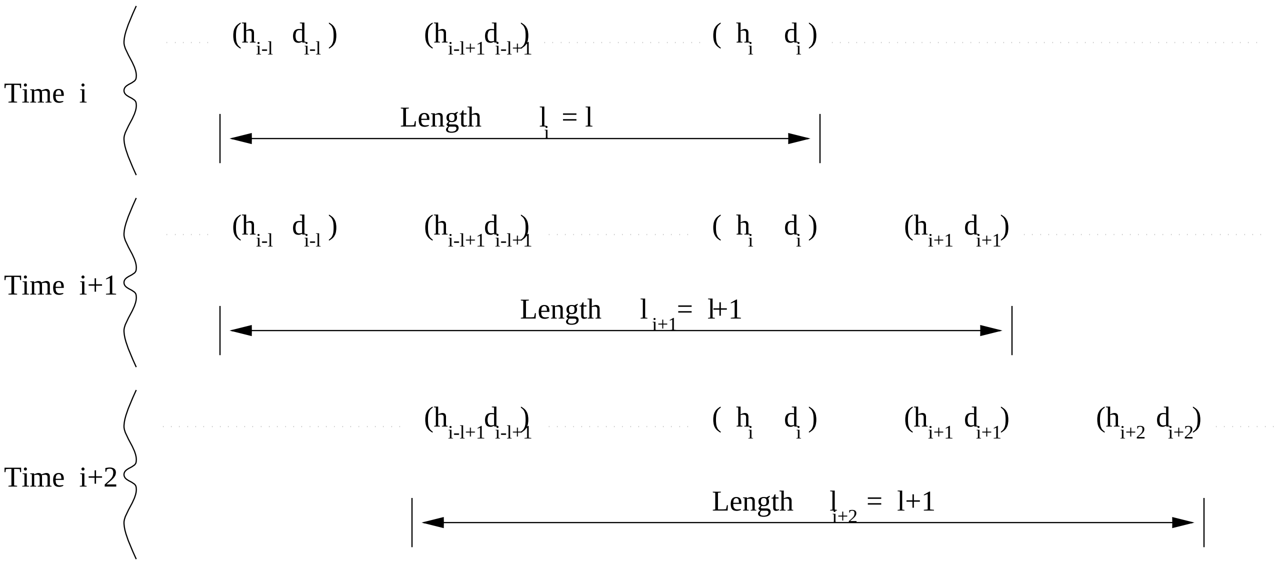


Figure 4.2: Sliding window with varying window length.

time instant we add the data point $\{h_{i+1}, d_{i+1}\}$, so that the window length changes to $l_{i+1} = l + 1$. At time $i + 2$ we add the data point $\{h_{i+2}, d_{i+2}\}$ and drop the data point $\{h_{i-l}, d_{i-l}\}$ so that the window length remains $l_{i+2} = l+1$. In a similar fashion, more general sliding window patterns can be considered as well.

To recast expression (4.4.1) into the usual quadratic form considered in this chapter, the lower index of the summation term needs to start at the fixed time 0. For this purpose, we rewrite $J_i$ as follows

$$J_i(w, d_{i-l_i+1}, \ldots, d_i, h_{i-l_i+1}, \ldots, h_i) = w^*\Pi_0^{-1}w + \sum_{j=i-l_i+1}^{i} (d_j - h_j w)^*(d_j - h_j w)$$

$$+\sum_{j=0}^{i-l_i}(d_j - h_j w)^*(d_j - h_j w) - \sum_{j=0}^{i-l_i}(d_j - h_j w)^*(d_j - h_j w) \qquad (4.4.2)$$

where we have added and subtracted identical terms. We now invoke a change of variables and substitute the time index $i$ by another time index $k$ that allows us to replace $J_i$ by a $\bar{J}_k$. The new index $k$ has the property whenever a new data point is added (*i.e.* , $i$ is incremented) then $k$ is incremented. However, whenever a data point is discarded from the window, $k$ is incremented as well. Thus if at time $i$ the length of the window is $l_i$, then the index $k$ will run from 0 to $2i - l_i + 1$ (since there will have been $i$ data points added and $i - l_i + 1$ data points removed). To be more specific, the change of variables is as follows.

(a) At each time $i$, since the data point $\{h_i, d_i\}$ is added, we increment the index $k$ and define

$$\bar{d}_k = d_i, \quad \bar{h}_k = h_i \quad \text{and} \quad \bar{R}_k = 1. \tag{4.4.3}$$

(b) If at time $i$ the data point $\{h_{i-l_i}, d_{i-l_i}\}$ is removed, we increment the index $k$ once more and define

$$\bar{d}_k = d_{i-l_i}, \quad \bar{h}_k = h_{i-l_i} \quad \text{and} \quad \bar{R}_k = -1. \tag{4.4.4}$$

With this convention we may write the quadratic form

$$J_i(w, d_{i-l_i+1}, \ldots, d_i, h_{i-l_i+1}, \ldots, h_i),$$

as

$$J_i = \bar{J}_k(w, d_0, \ldots, d_k, h_0, \ldots, h_k) = w^* \Pi_0^{-1} w + \sum_{j=0}^{k} (\bar{d}_j - \bar{h}_j w)^* (\bar{d}_j - \bar{h}_j w), \tag{4.4.5}$$

which is of the form that we have been considering in this thesis since Chapter 2. Note that the quadratic form $\bar{J}_k(w, d_0, \ldots, d_k, h_0, \ldots, h_k)$ is indefinite, since whenever a data point is dropped we have $\bar{R}_k = -1$. We can therefore use Krein space methods to solve the problem.

Using the same approach that we have used so far, we now construct the partially equivalent state-space model to the indefinite quadratic form $\bar{J}_k$. Thus

$$\begin{cases} \mathbf{x}_{j+1} &= \mathbf{x}_j, \quad \mathbf{x}_0 = \mathbf{w} \quad 0 \leq j \leq k \\ \bar{d}_j &= \bar{h}_j \mathbf{x}_k + \mathbf{v}_k \end{cases} \tag{4.4.6}$$

with $\Pi > 0$, $Q_j = 0$, $S_j = 0$ and $\bar{R}_j$ as in (4.4.3) and (4.4.4).

We can now state the following result.

**Theorem 4.4.1 (Finite Memory Adaptive Filter)** *The finite memory adaptive filter is given by the following recursions:*

*(a) For updating, the data point $\{h_i, d_i\}$ at time $i$ we have*

$$\hat{w}_{|i:i-l_{i-1}} = \hat{w}_{|i-1:i-l_{i-1}} + K_{p,k}(d_i - h_i\hat{w}_{|i-1:i-l_{i-1}}) \tag{4.4.7}$$

*where $\hat{w}_{|i:j}$ is the estimate when the sliding window encompasses all the data from time $j$ to time $i$, and*

$$K_{p,k} = P_k h_i^* R_{e,k}^{-1} \quad , \quad R_{e,k} = 1 + h_i P_k h_i^* \tag{4.4.8}$$

*and where $P_k$ satisfies the recursion*

$$P_{k+1} = P_k - K_{p,k} R_{e,k} K_{p,k}^*, \quad P_0 = \Pi_0. \tag{4.4.9}$$

*(b) For downdating, the data point $\{h_{i-l_i}, d_{i-l_i}\}$ at time $i$ we have*

$$\hat{w}_{|i:i-l_i+1} = \hat{w}_{|i:i-l_i} + K_{p,k}(d_{i-l_i} - h_{i-l_i}\hat{w}_{|i:i-l_i}) \tag{4.4.10}$$

*where*

$$K_{p,k} = P_k h_{i-l_i}^* R_{e,k}^{-1} \quad , \quad R_{e,k} = -1 + h_{i-l_i} P_k h_{i-l_i}^* \tag{4.4.11}$$

*and $P_k$ satisfies the recursion*

$$P_{k+1} = P_k - K_{p,k} R_{e,k} K_{p,k}^*. \tag{4.4.12}$$

*Moreover, the above solutions for $\hat{w}_{|i:j}$ always correspond to a minimum, and in particular*

$$R_{e,k} = 1 + h_i P_k h_i^* > 0, \tag{4.4.13}$$

*when we are updating, and*

$$R_{e,k} = -1 + h_{i-l_i} P_k h_{i-l_i}^* < 0, \tag{4.4.14}$$

*when we are downdating.*

**Proof:** The solutions given by (a) and (b) in the above Theorem are simply the Krein space Kalman filter recursions for the state-space model (4.4.6) which we know computes the stationary point of $\bar{J}_k$ over $w$. However, this stationary point is always a minimum since

$$\frac{\partial^2 \bar{J}_k}{\partial w^2} = \frac{\partial^2 J_i}{\partial w^2} = \Pi_0^{-1} + \sum_{j=i-l_i+1}^{i} h_j^* h_j > 0,$$

(recall that $\Pi_0^{-1} > 0$).

Using Lemma 2.7.3, having a minimum means that $R_{e,k}$ and $R_k$ have the same inertia for all $k$. Thus the statements (4.4.13) and (4.4.14) readily follow.

■

The fact that whenever we drop data we have $R_{e,k} < 0$ has an interesting interpretation. Consider the equation

$$P_{k+1} = P_k - K_{p,k} R_{e,k} K_{p,k}$$

If we drop data at step $k$ we would expect $P_{k+1}$ to get larger (more positive-definite) than $P_k$. This can only happen if $R_{e,k} < 0$. Thus, we may infer that innovations with negative Gramian correspond to a loss of information.

The above discussion puts the problem of finite memory adaptive filtering into the same state-space estimation framework as conventional adaptive filtering techniques (see [SK94b]). Therefore the various algorithmic extensions discussed there may be applied to finite memory problems, albeit that we now need to consider a Krein space. We shall not give the details here, but shall just mention that when the elements of the input vectors $\{h_i\}$ form a time sequence, viz.,

$$h_i = \begin{bmatrix} u_i & u_{i-1} & \ldots & u_{i-n+1} \end{bmatrix},$$

and when the window length is constant, *i.e.* , $l_i \equiv l$, then the state-space model (4.4.6) is periodic with period $T = 2$, and we may speed up the the estimation algorithm by a so-called Chandrasekhar-type recursion. Similar results, obtained via a different approach, have been reported in [Hou92].

## 4.5 Conclusion

In this chapter we studied the problems of risk-sensitive estimation and control, quadratic game theory, and finite memory adaptive filtering, using the framework of linear estimation in Krein spaces that was developed in Chapter 2. The theme that unites all these problems is that they can be related, albeit in an indirect fashion, to certain indefinite quadratic forms. We anticipate that there should be other problems that are amenable to this approach, and mention, in passing, an interesting application to recursive algorithms for (suboptimal) total least-squares problems that has been given in [SHK96a].