Chapter 7

Infinite Horizon Results

In this chapter we study the celebrated discrete-time algebraic Riccati equation (DARE) which arises in an impressive range of applications in systems and control theory. Although a great deal is known about the Riccati equation when the coefficient matrices are positive semi-definite, much less is known when these coefficients are indefinite matrices. Here we shall consider the DARE in the full generality of this, so-called, indefinite case and shall then particularize the results to some important special cases.

The approach taken in this chapter is given by the introduction of a certain (so-called) Popov function whose factorizations are intimately related to solutions of the DARE. The main result is that solutions to the DARE, or more more precisely a system of algebraic Riccati equations (SDARE), exists if, and only if, a certain proper factorization of the Popov function exists. Additional conditions are then given under which the solution to the DARE becomes stabilizing, Hermitian, positive semi-definite, etc. We also relate the solutions of the DARE to a so-called Hamiltonian matrix, and remark that the famous invariant subspace method can be used to compute solutions of the DARE in the indefinite case as well. Some examples are also included to illustrate the significance of the results.

7.1 Introduction

Undoubtedly one of the most important concepts in linear systems and control, both from a practical and a theoretical point of view, is the algebraic Riccati equation. Although the origin of the Riccati equation goes back to Count Riccati in 1724 [Ric24], it was first introduced into control theory by Kalman in 1960 [Kal60a]. Since then the algebraic Riccati equation has known an impressive range of applications such as linear quadratic (LQ) optimal control [AM71, KS72], stability theory [Kal63a, Yak62, Pop64, Wil76], stochastic filtering and stochastic control [Jaz70, AM79, Kai81, Ast70, BS78], stochastic realization theory [And69, Fau76, Pic76], synthesis of linear passive networks [KB61, AV73], dynamic games [Isa65, BO82] and, most recently, H^∞ optimal control and filtering [GD88, DGKF89, KN91, GL95].

There currently exists a vast and growing literature on the algebraic Riccati equation. (The survey papers [SH83, Dor83, Sha83, And88, LR91, Kuc91, Lau91] and the monograph [LR95] provide a good perspective of the research activity in this field.) However, most of the results available concern the special case where the coefficient matrices of the Riccati equation are positive semi-definite. This is quite natural because the Riccati equations encountered in LQ optimal control and filtering (which is where these equations first appeared) are of this, so-called positive, type. However, the Riccati equations that arise in the more recent fields of H^∞ and risk-sensitive (see *e.g.*, [Whi90]) optimal control and filtering are of the type that have (possibly) indefinite coefficient matrices. Therefore there is now growing interest in studying Riccati equations of this latter indefinite type.

In this chapter we shall be interested in the discrete-time algebraic Riccati equation in the general case where the coefficient matrices are possibly indefinite. Although we shall confine our attention here to the discrete-time case, one should be able to generalize the results to the continuous-time algebraic Riccati equation (CARE) and its associated Riccati differential equation.

Earlier approaches to studying the algebraic Riccati equation in the indefinite case essentially generalize the idea of invariant subspaces of the Hamiltonian matrix, first introduced by Potter [Pot66]. Here we shall take a different approach to studying the indefinite DARE that follows the line of the earlier chapters of this thesis on generalizing linear estimation theory to indefinite metric spaces (see Chapter 2). In order to do so, we shall introduce a so-called Popov function, that can be regarded as the generalization of the usual power spectral density function to indefinite metric spaces. In another sense, the Popov function can be regarded as a generalization of the concept of a "supply rate", introduced and used by Willems to study the positive DARE [Wil71b, Wil72], to indefinite metric spaces. In our framework, the Popov function is a natural object to study since its factorizations are intimately related to solutions of the DARE. Therefore, we can replace the problem of studying solutions to the DARE with the problem of finding factorizations of the Popov function. This latter problem can be studied by considering the Smith-Mcmillan canonical form of the Popov function, or by other means.

The major result of the chapter is that solutions to the DARE, or more more precisely a system of algebraic Riccati equations (SDARE), exists if, and only if, a certain proper factorization of the Popov function exists. Additional conditions are then given under which the solution to the DARE becomes stabilizing, Hermitian, positive semi-definite, etc.

The remainder of the chapter is organized as follows. In Sec. 7.2 we begin the study of the DARE by motivating the Popov function. We then proceed to establish some properties of the Popov function in Sec. 7.3. We first establish an equivalent class for the center matrices that yield the same Popov function and then study factorizations of the Popov function via the Smith-Mcmillan form. In particular, we define the notions of proper, proper canonical, and proper canonical Hermitian factorizations of the Popov function. These are then used to prove general existence results for the DARE in Sec. 7.4. These results essentially state that, solutions to the SDARE (with various properties) exist if, and only if, certain proper factorizations of the Popov function (with various properties) exist. Sec. 7.5 considers three special cases: the positive DARE, the case where the state matrix is invertible, and the case that frequently arises in H^∞ problems. Sec. 7.6 then introduces the Hamiltonian matrix and shows that the wellknown method of invariant subspaces, for finding solutions to the DARE, can be generalized to the indefinite case. Some examples that illustrate the concepts presented are given in Sec. 7.7, and the chapter concludes with Sec. 7.8.

7.2 The Discrete-time Algebraic Riccati Equation

In this section we shall study the properties of the discrete-time algebraic Riccati equation (DARE),

$$P = FPF^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^* + GQG^*, \qquad (7.2.1)$$

where $F \in \mathcal{C}^{n\times n}$, $G \in \mathcal{C}^{n\times m}$ and $H \in \mathcal{C}^{p\times n}$ are given, $Q = Q^* \in \mathcal{C}^{m\times m}$, $R = R^* \in \mathcal{C}^{p\times p}$ and $S \in \mathcal{C}^{m\times p}$ are known, and the unknown variable is the (possibly) Hermitian matrix, $P \in \mathcal{C}^{n\times n}$. Moreover, it is assumed that the Hermitian matrices

$$R \quad \text{and} \quad \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix},$$

are nonsingular.

We shall often find it convenient to introduce the notations

$$K_p = (FPH^* + GS)(R + HPH^*)^{-1} \quad \text{and} \quad R_e = R + HPH^*, \qquad (7.2.2)$$

and, when P is Hermitian, to rewrite the DARE (7.2.1) as

$$P = FPF^* - K_p R_e K_p^* + GQG^*. \qquad (7.2.3)$$

When, P, the solution to the DARE is not Hermitian, we can define

$$K_q = (FP^*H^* + GS)(R + HP^*H^*)^{-1}, \qquad (7.2.4)$$

and rewrite the DARE as

$$P = FPF^* - K_p R_e K_q^* + GQG^*. \qquad (7.2.5)$$

Solutions to the DARE turn out to be intimately related to factoring the following so-called *Popov function*,

$$\Sigma(z) = \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \qquad (7.2.6)$$

The connection comes through the fact that for *any* $n \times n$ matrix, Z, we can write:

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* - Z + FZF^* & GS + FZH^* \\ S^*G^* + HZF^* & R + HZH^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{7.2.7}$$

Now if we choose $Z = P$, a solution to the DARE, the center matrix in (7.2.7) drops rank, *i.e.*,

$$\begin{bmatrix} GQG^* - Z + FZF^* & GS + FZH^* \\ S^*G^* + HZF^* & R + HZH^* \end{bmatrix} = \begin{bmatrix} K_pR_eK_q^* & K_pR_e \\ R_eK_q^* & R_e \end{bmatrix} = \begin{bmatrix} K_p \\ I \end{bmatrix} R_e \begin{bmatrix} K_q^* & I \end{bmatrix},$$

where we have made use of (7.2.2), (7.2.4) and (7.2.5). Thus, it is straightforward to see that the Popov function (7.2.7) can be written as

$$\Sigma(z) = [H(zI-F)K_p + I]\, R_e \left[H(z^{-*}I - F)K_q + I\right]^*, \tag{7.2.8}$$

which is the factorization we claimed earlier.

Remark: If, for future reference, we define the center matrix in (7.2.7) as

$$N(Z) = \begin{bmatrix} GQG^* - Z + FZF^* & GS + FZH^* \\ S^*G^* + HZF^* & R + HZH^* \end{bmatrix}, \tag{7.2.9}$$

we may note that the DARE given by (7.2.1) is simply obtained by setting the *Schur complement* of the $(2,2)$ block entry in $N(Z)$ equal to zero. In this sense, we shall often refer to (7.2.1) as being the DARE corresponding to the Popov function $\Sigma(z)$, or similarly the DARE corresponding to the matrix $N(Z)$.

7.3 Properties of the Popov Function

With this motivation of the Popov function, we will now begin to establish some of its properties. These properties will be useful in our study of the DARE. But first, we remark that we shall henceforth implicitly assume that F is stable. However, in all the results given below we have replaced the stability condition on F with the weaker condition that $\{F, H\}$ is detectable[1]. The reason why this can be done is given below.

[1] There are various characterizations of detectability. The one that we shall use is that $\{F, H\}$ is detectable if, and only, if there exists a constant matrix K such that $F - KH$ is stable.

Lemma 7.3.1 (Detectable $\{F, H\}$) *When $\{F, H\}$ is detectable, the Popov function $\Sigma(z)$ can be written as*

$$\Sigma(z) = \left(I + H(zI - F)^{-1}K\right)\Sigma_k(z)\left(I + K^*(z^{-1}I - F^*)^{-1}H^*\right), \tag{7.3.1}$$

where

$$\Sigma_k(z) = \begin{bmatrix} H(zI - F + KH)^{-1} & I \end{bmatrix} N_k(Z) \begin{bmatrix} (z^{-1}I - F^* + H^*K^*)^{-1} \\ I \end{bmatrix}, \tag{7.3.2}$$

and where K is a constant matrix such that $F - KH$ is stable. Moreover $N_k(Z)$ is given by

$$N_k(Z) =$$
$$\begin{bmatrix} -Z + (F-KH)Z(F-KH)^* + \begin{bmatrix} G & -K \end{bmatrix}\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}\begin{bmatrix} G^* \\ -K^* \end{bmatrix} & (F-KH)ZH^* + \begin{bmatrix} G & -K \end{bmatrix}\begin{bmatrix} S \\ R \end{bmatrix} \\ HZ(F-KH)^* + \begin{bmatrix} S^* & R \end{bmatrix}\begin{bmatrix} G^* \\ -K^* \end{bmatrix} & R + HZH^* \end{bmatrix} \tag{7.3.3}$$

where Z is any $n \times n$ matrix. Finally, the discrete-time algebraic Riccati equation corresponding to $N_k(Z)$ is the same as the original DARE given by (7.2.1).

Remark: The point of Lemma 7.3.1 is that when $\{F, H\}$ is detectable we can always work instead with the Popov function $\Sigma_k(z)$ whose system matrix, $F - KH$, is stable. Thus, there is no loss of generality in assuming that F is stable, as long as we have the appropriate detectability condition.

Proof of Lemma 7.3.1: Since $\{F, H\}$ is detectable, we can choose a constant gain matrix, K, such that $F - KH$ is stable. Now we may write

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix}\begin{bmatrix} I & K \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} N(Z) \begin{bmatrix} I & 0 \\ -K^* & I \end{bmatrix}\begin{bmatrix} I & 0 \\ K^* & I \end{bmatrix}\begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

for any $Z \in \mathcal{C}^{n\times n}$. But some algebra shows that

$$\begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} N(Z) \begin{bmatrix} I & 0 \\ -K^* & I \end{bmatrix} = N_k(Z),$$

and likewise that

$$\begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix}\begin{bmatrix} I & K \\ 0 & I \end{bmatrix} = \left(I + H(zI-F)^{-1}K\right)\begin{bmatrix} H(zI - F + KH)^{-1} & I \end{bmatrix},$$

which yield (7.3.1), (7.3.2) (7.3.3).

To show the last statement, using some algebra, it is straightforward to show that the original DARE given by (7.2.1) can be rewritten as

$$P = (F-KH)P(F-KH)^* + \begin{bmatrix} G & -K \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} G^* \\ -K^* \end{bmatrix} - (K_p - K)R_e(K_q - K)^*.$$

But this is simply the DARE associated with $N_k(Z)$. (This can be seen by setting the Schur complement of the $(2,2)$ entry in $N_k(Z)$ equal to zero — see the remark preceding Sec. 7.3.)

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The next result concerns the nonuniqueness of the center matrices in (7.2.6) that yield the same Popov function, $\Sigma(z)$.

Lemma 7.3.2 (Equivalence Class for Center Matrices) *(a) For any $n \times n$ matrix Z, the Popov function,*

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

is invariant under the transformation

$$\begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \to \begin{bmatrix} GQG^* - Z + FZF^* & GS + FZH^* \\ S^*G^* + HZF^* & R + HZH^* \end{bmatrix}. \tag{7.3.4}$$

(b) Assume that the pair $\{F, H\}$ is detectable. If there exist Hermitian center matrices

$$\begin{bmatrix} G_1Q_1G_1^* & G_1S_1 \\ S_1^*G_1^* & R_1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} G_2Q_2G_2^* & G_2S_2 \\ S_2^*G_2^* & R_2 \end{bmatrix}$$

that yield the same Popov function, i.e.,

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} G_1Q_1G_1^* & G_1S_1 \\ S_1^*G_1^* & R_1 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \\ &= \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} G_2Q_2G_2^* & G_2S_2 \\ S_2^*G_2^* & R_2 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \end{aligned}$$

then there exists a <u>*unique*</u> $n \times n$ *Hermitian matrix,* Z*, such that*

$$\begin{bmatrix} G_1Q_1G_1^* & G_1S_1 \\ S_1^*G_1^* & R_1 \end{bmatrix} = \begin{bmatrix} G_2Q_2G_2^* - Z + FZF^* & G_2S_2 + FZH^* + V \\ S_2^*G_2^* + HZF^* + V^* & R_2 + HZH^* \end{bmatrix}, \tag{7.3.5}$$

where V *lies in the null-space of the observability matrix of* $\{F, H\}$*. In particular, if* $\{F, H\}$ *is observable, then there exists a unique Hermitian* Z *such that*

$$\begin{bmatrix} G_1Q_1G_1^* & G_1S_1 \\ S_1^*G_1^* & R_1 \end{bmatrix} = \begin{bmatrix} G_2Q_2G_2^* - Z + FZF^* & G_2S_2 + FZH^* \\ S_2^*G_2^* + HZF^* & R_2 + HZH^* \end{bmatrix}. \tag{7.3.6}$$

Remarks:

(i) Note that Lemma 7.3.2 parametrizes the nonuniqueness of the center matrices that yield the same Popov function, in terms of an arbitrary $Z \in \mathcal{C}^{n\times n}$. It is this freedom in choosing the parameter Z that allows us to reduce the rank of the center matrix, $N(Z)$, and thereby obtain a factorization of $\Sigma(z)$, via solutions to the DARE.

(ii) The above Lemma can be proven and interpreted by introducing linear time-invariant systems driven by inputs that lie in an indefinite, or so-called *Krein*, space. [This was actually done in Sec. 2.2.2.] However, we shall not adopt this route here, and shall give a different proof to Lemma 7.3.2.

Proof of Lemma 7.3.2: The proof of part (a) is via a direct calculation — simply check that the identity

$$\begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} -Z + FZF^* & FZH^* \\ HZF^* & HZH^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} = 0,$$

holds for any $Z \in \mathcal{C}^{n\times n}$.

To prove part (b), we proceed as follows. First, note that since we have assumed $\{F, H\}$ detectable, using Lemma 7.3.1, there is no loss of generality in assuming that

F is stable. With F stable let us chose Z_1 and Z_2 as the unique (Hermitian) solutions of the following Lyapunov equations[2]

$$G_1Q_1G_1^* - Z_1 + FZ_1F^* = 0 \quad \text{and} \quad G_2Q_2G_2^* - Z_2 + FZ_2F^* = 0. \tag{7.3.7}$$

With these choices of Z_1 and Z_2, and using the result of part (a), we may write

$$\begin{aligned}
\Sigma(z) &= \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & FZ_1H^* + G_1S_1 \\ HZ_1F^* + S_1^*G_1^* & R_1 + HZ_1H^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \\
&= \sum_{j=1}^{\infty} HF^{j-1}(FZ_1H^* + G_1S_1)z^{-j} + (R_1 + HZ_1H^*) + \sum_{j=1}^{\infty}(FZ_1H^* + G_1S_1)^*F^{*(j-1)}H^*z^j,
\end{aligned}$$

and

$$\begin{aligned}
\Sigma(z) &= \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & FZ_2H^* + G_2S_2 \\ HZ_2F^* + S_2^*G_2^* & R_2 + HZ_2H^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \\
&= \sum_{j=1}^{\infty} HF^{j-1}(FZ_2H^* + G_2S_1)z^{-j} + (R_2 + HZ_2H^*) + \sum_{j=1}^{\infty}(FZ_2H^* + G_2S_2)^*F^{*(j-1)}H^*z^j,
\end{aligned}$$

where we have also used the fact that F is stable to perform the above expansions.

Equating the coefficients of z^j in the above two expansions for $\Sigma(z)$, yields

$$R_1 + HZ_1H^* = R_2 + HZ_2H^*, \tag{7.3.8}$$

and

$$HF^{j-1}(FZ_1H^* + G_1S_1) = HF^{j-1}(FZ_2H^* + G_2S_1), \quad j \geq 1.$$

The last of the above two equalities implies that

$$\mathcal{O}(FZ_1H^* + G_1S_1) = \mathcal{O}(FZ_2H^* + G_2S_1),$$

where $\mathcal{O}$ is the observability matrix corresponding to the pair $\{F, H\}$. Thus, $FZ_1H^* + G_1S_1$ and $FZ_2H^* + G_2S_1$ differ by some matrix V in the null-space of $\mathcal{O}$, and we may write

$$FZ_1H^* + G_1S_1 = FZ_2H^* + G_2S_1 + V. \tag{7.3.9}$$

[2] In fact, for the solution of the Lyapunov equation $GQG^* - Z + FZF^* = 0$ to be unique, we only require that F have no two eigenvalues such that $\lambda_i = \lambda_j^{-*}$ — see *e.g.*, [Kai80].

Combining (7.3.7)-(7.3.9) yields

$$\begin{bmatrix} G_1Q_1G_1^* & G_1S_1 \\ S_1^*G_1^* & R_1 \end{bmatrix} =$$

$$\begin{bmatrix} G_2Q_2G_2^* - (Z_2 - Z_1) + F(Z_2 - Z_1)F^* & G_2S_2 + F(Z_2 - Z_1)H^* + V \\ S_2^*G_2^* + H(Z_2 - Z_1)F^* + V^* & R_2 + H(Z_2 - Z_1)H^* \end{bmatrix},$$

so that defining the unique matrix $Z = Z_2 - Z_1$ proves (7.3.5).

To prove (7.3.6) we note that when $\{F, H\}$ is observable, $\mathcal{O}$ has full rank, which means $V = 0$ and leads to the desired result.

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A very similar result holds when the center matrix in the Popov function is not restricted to be Hermitian. Since we shall make use of this result, we state it here as a corollary. The proof is exactly the same as the one given above and is therefore not repeated.

Corollary 7.3.1 (Non-Hermitian Center Matrices) *Assume that the pair $\{F, H\}$ is detectable. If there exist (possibly non-Hermitian) center matrices*

$$\begin{bmatrix} Q_1 & S_{a,1} \\ S_{b,1}^* & R_1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} Q_2 & S_{a,2} \\ S_{b,2}^* & R_2 \end{bmatrix}$$

that yield the same Popov function, i.e.,

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_1 & S_{a,1} \\ S_{b,1}^* & R_1 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \\ &= \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_2 & S_{a,2} \\ S_{b,2}^* & R_2 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \end{aligned}$$

then there exists a <u>*unique*</u> *$n \times n$ matrix, Z, such that*

$$\begin{bmatrix} Q_1 & S_{a,1} \\ S_{b,1}^* & R_1 \end{bmatrix} = \begin{bmatrix} Q_2 - Z + FZF^* & S_{a,2} + FZH^* + V_a \\ S_{b,2}^* + HZF^* + V_b^* & R_2 + HZH^* \end{bmatrix}, \tag{7.3.10}$$

where V_a and V_b lie in the null-space of the observability matrix of $\{F, H\}$. In particular, if $\{F, H\}$ is observable, then there exists a unique Z such that

$$\begin{bmatrix} Q_1 & S_{a,1} \\ S_{b,1}^* & R_1 \end{bmatrix} = \begin{bmatrix} Q_2 - Z + FZF^* & S_{a,2} + FZH^* \\ S_{b,2}^* + HZF^* & R_2 + HZH^* \end{bmatrix}. \tag{7.3.11}$$

7.3.1 Factorization of the Popov Function

Our next result concerns factorizations of the Popov function, $\Sigma(z)$. We shall essentially use the Smith-Mcmillan form of $\Sigma(z)$ to study the existence of various forms of factorization for $\Sigma(z)$. When coupled with Lemma 7.3.2, this result will allow us to give conditions for the existence of solutions to the DARE, in terms of the existence of certain factorizations, and to establish their various properties.

Lemma 7.3.3 (Factorization of the Popov Function) *Consider the Popov function*

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I-F^*)^{-1}H^* \\ I \end{bmatrix},$$

where $\{F, H\}$ is detectable. Then $\Sigma(z)$ can always be factorized in the fashion

$$\Sigma(z) = M(z)N^*(z^{-*}), \tag{7.3.12}$$

where $M(z)$ and $N(z)$ are $p \times q$ $(p \leq q)$ rational transfer matrices that have poles of magnitude less than one and zeros of magnitude less than or equal to one.

Proof: The proof basically relies on the Smith-McMillan form (see *e.g.*, [McM52, Kai80]) of the Popov function $\Sigma(z)$. Since the Popov function is para-Hermitian, *i.e.*, $\Sigma(z) = \Sigma^*(z^{-*})$, then using elementary row and column operations, the Smith form of $\Sigma(z)$ may be written as

$$\Sigma(z) = U(z)\Lambda(z)V^*(z^{-*}),$$

where $U(z)$ and $V(z)$ are unimodular (here this means that both $U(z)$ and $U^{-1}(z)$ — likewise, $V(z)$ and $V^{-1}(z)$ — are polynomial matrices in z) and where the diagonal matrix

$$\Lambda(z) = \begin{bmatrix} \frac{b_1(z)}{a_1(z)} & & & & & \\ & \ddots & & & & \\ & & \frac{b_q(z)}{a_q(z)} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix},$$

is such that

$$a_i(z)|a_{i+1}(z) \quad (i.e., \ a_i(z) \text{ divides } a_{i+1}(z)), \quad \text{and} \quad b_{i+1}(z)|b_i(z), \quad i = 1, \ldots, q-1.$$

The roots of the polynomials $a_i(z)$ and $b_i(z)$ are the poles and zeros of $\Sigma(z)$, respectively.

Now note that, since $\Sigma(z)$ is a para-Hermitian, for every pole (or zero) of $\Sigma(z)$ at $z = \alpha$ there exists a pole (correspondingly, zero) at $z = \alpha^{-*}$. Due to the observability condition, we may assume that F is stable which means that none of the poles of $\Sigma(z)$ (*i.e.*, the eigenvalues of F and F^{-*}) lie on the unit circle. However, zeros on the unit circle may exist. Moreover, such zeros need not appear in pairs, since

$$\alpha = e^{j\omega_0} = 1/e^{-j\omega_0} = \alpha^{-*}.$$

With these observations, we can factorize the diagonal entries of $\Lambda(z)$ as

$$a_i(z) = a_{m,i}(z)a^*_{m,i}(z^{-*}) \quad \text{and} \quad b_i(z) = b_{m,i}(z)b^*_{n,i}(z^{-*}), \quad i = 1, \ldots q,$$

where $a_{m,i}(z)$ is a polynomial in z with roots of magnitude less than one, and $b_{m,i}(z)$ and $b_{n,i}(z)$ are polynomials in z with roots of magnitude less than or equal to one.

Using the aforementioned Smith-McMillan form, along with the above factorization of $\Lambda(z)$, allows us to write $\Sigma(z)$ as

$$\Sigma(z) = U(z) \underbrace{\begin{bmatrix} \frac{b_{m,1}(z)}{a_{m,1}(z)} & & & \\ & \ddots & & \\ & & \frac{b_{m,q}(z)}{a_{m,q}(z)} & \\ & & & \bigcirc \end{bmatrix}}_{\left[\begin{array}{cc} M(z) & 0 \end{array}\right]} \underbrace{\begin{bmatrix} \frac{b^*_{n,1}(z^{-*})}{a^*_{m,1}(z^{-*})} & & & \\ & \ddots & & \\ & & \frac{b^*_{n,q}(z^{-*})}{a^*_{m,q}(z^{-*})} & \\ & & & \bigcirc \end{bmatrix}}_{\left[\begin{array}{c} N^*(z^{-*}) \\ 0 \end{array}\right]} U^*(z^{-*}),$$

where we have indicated the rational transfer matrices $M(z)$ and $N(z)$. Note that both $M(z)$ and $N(z)$ are of dimension $p \times q$ and that they have poles of magnitude less than one and zeros of magnitude less than or equal to one (since $U(z)$ and $V(z)$ do not have any poles or zeros). This establishes the claim of the lemma.

■

The above result shows that under a certain detectability assumption the Popov function always admits a certain (stable-antistable) factorization. Since at the beginning of Sec. 7.2.1 we noted the connection between factorizations of the Popov function and solutions to the DARE, we may now suspect Lemma 7.3.3 to have implications to the DARE. Unfortunately, the situation is not quite as simple, and, in fact, to relate to the DARE, what is needed is not any factorization of the Popov function, but certain (so-called) proper factorizations. These factorizations, along with several others, are defined below. It turns out that, contrary to the factorization of Lemma 7.3.3, these factorizations do not necessarily exist, and that their existence is tied to the existence of certain solutions to the DARE.

Definition 7.3.1 (Factorizations of the Popov Function) *Consider the Popov function*

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

where $\{F, H\}$ is detectable.

(i) $\Sigma(z)$ *is said to have a "proper" factorization if, we can write,*

$$\Sigma(z) = M(z)N^*(z^{-*}), \tag{7.3.13}$$

where $M(z)$ *and* $N(z)$ *are* $p \times q$ $(p \leq q)$ *proper rational transfer matrices, with Mcmillan degree less than or equal to* n*, that have poles of magnitude less than one.*

- *(a) If, in addition,* $M(z)$ *and* $N(z)$ *have zeros of magnitude less than or equal to one, then (7.3.13) is said to be a proper "semi-canonical" factorization of* $\Sigma(z)$*.*
- *(b) If, in addition,* $M(z)$ *and* $N(z)$ *have zeros of magnitude strictly less one, then (7.3.13) is said to be a proper "canonical" factorization of* $\Sigma(z)$*.*

(ii) $\Sigma(z)$ *is said to have a proper "Hermitian" factorization if, we can write,*

$$\Sigma(z) = M(z)JM^*(z^{-*}), \tag{7.3.14}$$

where the $p \times q$ $(p \leq q)$ *proper rational transfer matrix* $M(z)$ *has Mcmillan degree less than or equal to* n*, and poles of magnitude less than one, and where* J *is a signature matrix (*i.e.*, a diagonal matrix with* ± 1 *on the main diagonal) that represents the inertia of* $\Sigma(z)$*.*

- *(a) If, in addition,* $M(z)$ *has zeros of magnitude less than or equal to one, then (7.3.14) is said to be a proper "semi-canonical" Hermitian factorization of* $\Sigma(z)$*.*
- *(b) If, in addition,* $M(z)$ *has zeros of magnitude strictly less than one, then (7.3.14) is said to be a proper "canonical" Hermitian factorization of* $\Sigma(z)$*.*

The reason why we have introduced the above factorizations is that the factors given by the solutions to the DARE are proper (see Eq. (7.2.8)). On the other hand, the factors obtained from the Smith-McMillan form, *e.g.*,

$$\begin{bmatrix} M(z) & 0 \end{bmatrix} = U(z) \begin{bmatrix} \frac{b_{m,1}(z)}{a_{m,1}(z)} & & & \\ & \ddots & & \\ & & \frac{b_{m,q}(z)}{a_{m,q}(z)} & \\ & & & \bigcirc \end{bmatrix},$$

are not necessarily proper. [Note that, although the diagonal matrix on the RHS is proper, its product with the polynomial matrix, $U(z)$, need not be so.]

We should also mention that, for arbitrary para-Hermitian rational transfer matrices, $\Sigma(z)$, the conditions required for the existence of the above proper factorizations are not known.[3] Moreover, there are many other questions in the factorization of rational matrix functions, and matrix polynomials, that are open. For example, the question of whether a para-Hermitian matrix polynomial, $P(z)$, admits a Hermitian factorization,

$$P(z) = P^{1/2}(z) J P^{*/2}(z^{-*}), \tag{7.3.15}$$

with $P^{1/2}(z)$ polynomial[4] and J a signature matrix, is also open. A sufficient condition is due to Yakubovich [Yak70a], and is given below.

Lemma 7.3.4 (Hermitian Factorizations) *Suppose the multiplicity of the unit circle zeros of each of the invariant polynomials[5] of the para-Hermitian polynomial matrix $P(z)$ is even. Then $P(z)$ has a Hermitian factorization (7.3.15).*

The above condition is not necessary, as exemplified by the following factorization,

$$P(z) = \begin{bmatrix} e^{-j\frac{\theta_1-\theta_2}{2}}(z-e^{j\theta_1})(z^{-1}-e^{-j\theta_2}) & 0 \\ 0 & -e^{-j\frac{\theta_1-\theta_2}{2}}(z-e^{j\theta_1})(z^{-1}-e^{-j\theta_2}) \end{bmatrix}, \tag{7.3.16}$$

[3]In fact, in the next section we shall give such conditions when $\Sigma(z)$ can be represented by a Popov function.

[4]If $P^{1/2}(z)$ is not restricted to be polynomial the factorization (7.3.15) can always be achieved, say, by Hermitian Gaussian elimination, provided $P(z)$ has constant inertia almost everywhere on the unit circle.

[5]For the definition of the invariant polynomials of a polynomial matrix see [Kai80].

with

$$P^{1/2}(z) = \begin{bmatrix} z - e^{j\frac{\theta_1+\theta_2}{2}} & -(2+\cos\frac{\theta_1-\theta_2}{2})^{-1/2} \\ -(2+\cos\frac{\theta_1-\theta_2}{2})^{-1/2} & z - e^{-j\frac{\theta_1+\theta_2}{2}} \end{bmatrix} \tag{7.3.17}$$

and $J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Note that the $(z - e^{j\theta_1})(z^{-1} - e^{-j\theta_2})$'s, *i.e.*, the invariant polynomials of $P(z)$, have simple (odd) unit circle zeros.

All the aforementioned difficulties are due to the fact that $\Sigma(z)$ has been taken to be an arbitrary para-Hermitian rational transfer matrix. When $\Sigma(z)$ is positive semidefinite on the unit circle, *i.e.*,

$$\Sigma(e^{j\omega}) \geq 0, \quad \omega \in [0, 2\pi] \tag{7.3.18}$$

then we have the following result that is attributed to Youla [You61].

Theorem 7.3.1 (Factorization of Positive Rational Matrices) *Consider the proper $p \times p$ rational para-Hermitian matrix, $\Sigma(z)$, that is nonnegative definite on the unit circle. Then we may write,*

$$\Sigma(z) = M(z)M^*(z^{-*}), \tag{7.3.19}$$

where $M(z)$ is a $p \times q$ ($q \leq p$) proper rational matrix function with poles of magnitude less than one and zeros of magnitude less than or equal to one. If, in addition, $\Sigma(z)$ has constant rank (equal to q) everywhere on the unit circle, then all the zeros of $M(z)$ lie strictly inside the unit circle.

Proof: The proof is lengthy and will not be given here. The reader is referred to [You61] for the details. We just mention, in passing, that the proof uses the Smith-McMillan form of $\Sigma(z)$ along with a certain construction, due to Oono and Yasuura [OY54], for the factorization of positive-definite unimodular para-Hermitian matrices.[6]

∎

[6] In fact, a simple generalization of this ingenious construction has been used in [Yak70a] to study the factorization of indefinite para-Hermitian matrices.

Before closing this section we note that the factorization of Theorem 7.3.1 is a special case of the Hermitian factorization (7.3.14) when $J = I$. Moreover, in this positive case, we cannot have nonproper factors, since if $M(z)$ were nonproper then so would be the product, $M(z)M^*(z^{-*})$. However, this is no longer true in the indefinite case — we can still have nonproper factors, since the signature matrix J in $M(z)JM^*(z^{-*})$ could allow for cancellations that render the product proper.

7.4 A General Existence Result

We now present the main result of this chapter that essentially gives necessary and sufficient conditions for the existence of a solution to the DARE (7.2.1) in terms of the existence of certain proper factorizations of the associated Popov function.

Theorem 7.4.1 (Existence of Solutions to the DARE) *Consider the system of discrete-time algebraic Riccati equations (SDARE)*

$$\left\{\begin{array}{rcl} P &=& FPF^* + GQG^* - K_p R_e K_q^* \\ K_p R_e &=& FPH^* + GS \quad , \quad K_q R_e^* = FP^*H^* + GS \\ R_e &=& R + HPH^* \end{array}\right. \tag{7.4.1}$$

where $\{F, H\}$ is detectable. Then we have the following results.

(a) The SDARE (7.4.1) has a solution, P, if, and only if, the Popov function

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

admits a proper factorization of the form

$$\Sigma(z) = M(z) R_e N^*(z^{-*}),$$

where $M(z)$ and $N(z)$ are $p \times p$ proper rational transfer matrices, with Mcmillan degree less than or equal to n, that have poles of magnitude less than one, and where R_e is a constant matrix that yields the normalizations, $M(\infty) = N(\infty) = I_p$. Moreover, we have

$$M(z) = H(zI-F)^{-1}K_p + I_p \quad \textit{and} \quad N(z) = H(zI-F)^{-1}K_q + I_p.$$

(b) *The SDARE (7.4.1) has a "semi-stabilizing" solution, P,* i.e., *one for which,*

$$F - K_p H \quad \textit{and} \quad F - K_q H,$$

are semi-stable,[7] *if, and only if, $\Sigma(z)$ admits a proper semi-canonical factorization,* i.e., *one for which $M(z)$ and $N(z)$ have, in addition, zeros of magnitude less than or equal to one. Moreover, the solution is stabilizing,* i.e., *$F - K_p H$ and $F - K_q H$ are both stable, if, and only if, $\Sigma(z)$ admits a proper canonical factorization.*

(c) *The SDARE (7.4.1) has a Hermitian solution, P, if, and only if, the Popov function admits a proper Hermitian factorization,*

$$\Sigma(z) = M(z) R_e M^*(z^{-*}),$$

where $M(z)$ is a $p \times p$ proper rational transfer matrix with Mcmillan degree less than or equal to n and poles of magnitude less than one, and where R_e is a constant Hermitian matrix that yields the normalization, $M(\infty) = I_p$. In this case, we have

$$M(z) = H(zI - F)^{-1} K_p + I_p.$$

(d) *The SDARE (7.4.1) has a (semi-)stabilizing Hermitian solution, P, (*i.e., *$F - K_p H$ is (semi)stable), if, and only if, the Popov function admits a proper (semi-)canonical Hermitian factorization (*i.e., *$M(z)$ has, in addition, zeros of magnitude less than (or equal to) one).*

(e) *The SDARE has a solution, P, such that $R + HPH^*$ is nonsingular, if, and only if, the Popov function $\Sigma(z)$ is nonsingular a.e.*[8] *on the unit circle, $|z| = 1$, and admits a proper factorization. In this case, P is a solution to the DARE*

$$P = FPF^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^* + GQG^*.$$

---

[7] We shall call a matrix semistable if all its eigenvalues have magnitude less than or equal to one.

[8] Almost everywhere.

Proof: We begin with the proof of part (a). We have already seen that if a solution to the SDARE exists then $\Sigma(z)$ allows for a proper factorization (see the arguments leading to Eq. (7.2.8)). Therefore let us assume that a proper factorization (with the properties mentioned in the statement of the Theorem) exists. This means we may write

$$\Sigma(z) = M(z)N^*(z^{-*}) = A(z)DB^*(z^{-*}),$$

where $A(z)$ and $B(z)$ are $p \times p$ matrices with poles and zeros less than or equal to one, and D is a (possibly singular) $p \times p$ matrix that is chosen such that $A(z)$ and $B(z)$ satisfy the normalizations,

$$A(\infty) = I \quad \text{and} \quad B(\infty) = I.$$

Let us now construct minimal realizations for $A(z)$ and $B(z)$ as follows

$$A(z) = H_a(zI - F_a)^{-1}K_a + I \quad \text{and} \quad B(z) = H_b(zI - F_b)^{-1}K_b + I.$$

At this point we should remark that the zeros of $A(z)$ and $B(z)$ are given by the eigenvalues of $F_a - K_aH_a$ and $F_b - K_bH_b$, respectively. Therefore, since $A(z)$ and $B(z)$ have zeros of magnitude less than or equal to one, the matrices $F_a - K_aH_a$ and $F_b - K_bH_b$ are marginally (or semi) stable.

Now note that we may write the Popov function as

$$\begin{aligned}\Sigma(z) &= \left[H_a(zI - F_a)^{-1}K_a + I\right] D \left[H_b(z^{-*}I - F_b)^{-1}K_b + I\right]^* \\ &= \begin{bmatrix} H_a(zI - F_a)^{-1} & I \end{bmatrix} \begin{bmatrix} K_aDK_b^* & K_aD \\ DK_b^* & D \end{bmatrix} \begin{bmatrix} (z^{-1}I - F_b^*)^{-1}H_b^* \\ I \end{bmatrix}.\end{aligned}$$

Defining Z_{ab} as the unique solution to the Lyapunov equation,[9]

$$Z_{ab} = F_aZ_{ab}F_b^* + K_aDK_b^*, \tag{7.4.2}$$

and using the, readily verified, identity that

$$\begin{bmatrix} H_a(zI - F_a)^{-1} & I \end{bmatrix} \begin{bmatrix} -Z + F_aZF_b^* & F_aZH_b^* \\ H_aZF_b^* & H_aZH_b^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F_b^*)^{-1}H_b^* \\ I \end{bmatrix} = 0, \tag{7.4.3}$$

[9]Since both $A(z)$ and $B(z)$ are stable, the matrices F_a and F_b are stable as well. Therefore the Lyapunov equation will have a unique solution since F_a and F_b have no two eigenvalues, λ_a and λ_b, such that $\lambda_a = \lambda_b^{-*}$.

for any matrix Z of the appropriate dimensions, allows us to write the Popov function as,

$$\Sigma(z) = \begin{bmatrix} H_a(zI - F_a)^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & F_a Z_{ab} H_b^* + K_a D \\ H_a Z_{ab} F_b^* + D K_b^* & D + H_a Z_{ab} H_b^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F_b^*)^{-1} H_b^* \\ I \end{bmatrix}, \tag{7.4.4}$$

Now we can expand $\Sigma(z)$ using (7.4.4) to obtain

$$\Sigma(z) = \sum_{j=1}^{\infty} H_a F_a^{j-1}(F_a Z_{ab} H_b^* + K_a D) z^{-j} + (D + H_a Z_{ab} H_b^*) + \sum_{j=1}^{\infty} \left[H_b F_b^{j-1}(F_b Z_{ab}^* H_a^* + K_b D^*) \right]^* z^j.$$

A similar expansion for the Popov function in terms of its original representation (7.2.6) yields

$$\Sigma(z) = \sum_{j=1}^{\infty} H F^{j-1}(F Z H^* + GS) z^{-j} + (R + H Z H^*) + \sum_{j=1}^{\infty} \left[H F^{j-1}(F Z^* H^* + GS) \right]^* z^j,$$

where Z is the unique solution to the Lyapunov equation $Z = FZF^* + GQG^*$.

Equating the coefficients of the powers of z in these two expansions yields

$$H_a F_a^{j-1}(F_a Z_{ab} H_b^* + K_a D) = H F^{j-1}(F Z H^* + GS)$$

and

$$H_b F_b^{j-1}(F_b Z_{ab}^* H_a^* + K_b D^*) = H F^{j-1}(F Z H^* + GS),$$

for all $j \geq 1$. The above two relations mean that the system with state-space representation $\{F, FZH^* + GS, H\}$, has the same Markov parameters as the systems with state-space representation $\{F_a, F_a Z_{ab} H_b^* + K_a D, H_a\}$ and $\{F_b, F_b Z_{ab}^* H_a^* + K_b D^*, H_b\}$. Thus, there must exist a similarity transformation T_a such that

$$\begin{bmatrix} H_a & 0 & H_{a,c} & 0 \end{bmatrix} = H T_a \quad , \quad \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} = T_a^{-1} F T_a, \tag{7.4.5}$$

and

$$\begin{bmatrix} F_a Z_{ab} H_b^* + K_a D \\ N_{a,o} \\ 0 \\ 0 \end{bmatrix} = T_a^{-1}(F Z H^* + GS), \tag{7.4.6}$$

where the subscripts 'c', 'o' and 'co' represent the uncontrollable, the unobservable, and the both uncontrollable and unobservable, modes of the system $\{F, FZH^* + GS, H\}$ (see [Kai80], p. 133). Correspondingly, there exists a similarity transformation, T_b, relating $\{F, FZH^* + GS, H\}$ to $\{F_b, F_b Z_{ab}^* H_a^* + K_b D^*, H_b\}$.

On the other hand, $A(z)$ and $B(z)$ can also be written as

$$
\begin{aligned}
A(z) &= \begin{bmatrix} H_a & 0 & H_{a,c} & 0 \end{bmatrix} \left(zI - \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} \right)^{-1} \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} + I \\
&= H(zI - F)^{-1} T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} + I,
\end{aligned}
$$

and

$$
\begin{aligned}
B(z) &= \begin{bmatrix} H_b & 0 & H_{b,c} & 0 \end{bmatrix} \left(zI - \begin{bmatrix} F_b & 0 & F_{b,13} & 0 \\ F_{b,21} & F_{b,o} & F_{b,23} & F_{b,24} \\ 0 & 0 & F_{b,c} & 0 \\ 0 & 0 & F_{b,43} & F_{b,co} \end{bmatrix} \right)^{-1} \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix} + I \\
&= H(zI - F)^{-1} T_a \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix} + I,
\end{aligned}
$$

for *any* matrices X_a and X_b of the appropriate dimensions. Thus, we may write

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^* & T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \\ D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^* & D \end{bmatrix} \begin{bmatrix} (z^{-1}I-F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{7.4.7}$$

Now (7.4.7) and

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I-F^*)^{-1}H^* \\ I \end{bmatrix},$$

are two different representations of the Popov function. Therefore, using Corollary 7.3.1, there exists a unique P and V_a, V_b in the unobservable subspace of $\{F, H\}$ such that

$$\begin{bmatrix} -P+FPF^*+GQG^* & FPH^*+GS-V_a \\ HPF^*+S^*G^*-V_b^* & R+HPH^* \end{bmatrix} = \begin{bmatrix} T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^* & T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \\ D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^* & D \end{bmatrix}. \tag{7.4.8}$$

We shall now show that X_a and X_b can be chosen such that $V_a = 0$ and $V_b = 0$, respectively. To this end, first note that P is given by $P = Z - Z_1$, where

$$Z = FZF^* + GQG^*,$$

and

$$Z_1 = FZ_1F^* + T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^*. \tag{7.4.9}$$

In Appendix 7.A, we show that Z_1 has the form:

$$Z_1 = T_a \begin{bmatrix} Z_{ab} & Z_{ab,o} & 0 & 0 \\ Z_{ba,o} & Z_{ab,oo} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} T_b^*, \tag{7.4.10}$$

where

$$\begin{bmatrix} Z_{ab} & Z_{ab,o} \\ Z_{ba,o} & Z_{ab,oo} \end{bmatrix} = \begin{bmatrix} F_a & 0 \\ F_{a,21} & F_{a,o} \end{bmatrix} \begin{bmatrix} Z_{ab} & Z_{ab,o} \\ Z_{ba,o} & Z_{ab,oo} \end{bmatrix} \begin{bmatrix} F_b^* & F_{b,21}^* \\ 0 & F_{b,o}^* \end{bmatrix} + \begin{bmatrix} K_a \\ X_a \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \end{bmatrix}^*. \tag{7.4.11}$$

Equating the $(1,2)$ block entries in (7.4.8), we have

$$\begin{aligned} V_a &= FPH^* + GS - T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D = FZH^* + GS - FZ_1H^* - T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \\ &= T_a \begin{bmatrix} F_aZ_{ab}H_b^* + K_aD \\ N_{a,o} \\ 0 \\ 0 \end{bmatrix} - \\ & \quad T_a \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} \begin{bmatrix} Z_{ab} & Z_{ab,o} & 0 & 0 \\ Z_{ba,o} & Z_{ab,oo} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} H_b^* \\ 0 \\ H_{b,c}^* \\ 0 \end{bmatrix} - T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \end{aligned}$$

where in the second step we used the similarity transformation (7.4.6) to evaluate $FZH^* + GS$. Simplifying the above expression yields

$$V_a = T_a \begin{bmatrix} 0 \\ N_a - F_{a,21}Z_{ab}H_b^* - F_{a,o}Z_{ba,o}H_b^* - X_aD \\ 0 \\ 0 \end{bmatrix}.$$

Note that V_a belongs to the unobservable space of $\{F, H\}$. Using (7.4.11), $Z_{ba,o}$ satisfies the Lyapunov equation $Z_{ba,o} = F_{a,o}Z_{ba,o}F_b^* + F_{a,21}Z_{ab}F_b^* + X_aDK_b^*$. Moreover, since Z_{ab} satisfies (7.4.2) it does not depend on the free parameter X_a. Therefore, to demonstrate that X_a can be chosen such that $V_a = 0$, the following system of linear equations must be solvable in $Z_{ba,o}$ and X_aD

$$\begin{cases} N_a &= F_{a,o}Z_{ba,o}H_b^* + X_aD + F_{a,12}Z_{ab}H_b^* \\ Z_{ba,o} &= F_{a,o}Z_{ba,o}F_b^* + X_aDK_b^* + F_{a,12}Z_{ab}F_b^* \end{cases}.$$

But post-multiplying the first of the above equations by K_b^* and subtracting yields

$$Z_{ba,o} = F_{a,o}Z_{ba,o}(F_b - K_bH_b)^* + F_{a,12}Z_{ab}(F_b - K_bH_b)^* + N_aK_b^*.$$

This equation will have a unique solution in $Z_{ba,o}$, if $F_{a,o}$ and $F_b - K_bH_b$ have no two eigenvalues, $\lambda_{a,o}$ and λ_b, such that $\lambda_{a,o} = \lambda_b^{-*}$. But since $\lambda_{a,o}$ is an unobservable (stable) eigenvalue of F and λ_b is a zero of $B(z)$ we can assume this.[10] Thus, the unique solution for X_aD is $X_aD = N_a - F_{a,o}Z_{ba,o}H_b^* - F_{a,12}Z_{ab}H_b^*$. With this choice of X_a (and a similar choice for X_b), we have $V_a = 0$ (and similarly $V_b = 0$), and if we define

$$K_p = T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix}, \quad K_q = T_b \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad R_e = D,$$

Eq. (7.4.8) becomes

$$\begin{bmatrix} -P + FPF^* + GQG^* & FPH^* + GS \\ HPF^* + S^*G^* & R + HPH^* \end{bmatrix} = \begin{bmatrix} K_pR_eK_q^* & K_pR_e \\ R_eK_q^* & R_e \end{bmatrix},$$

[10]As mentioned earlier, since we have assumed the detectability of $\{F, H\}$, we may take F as stable, in which case due to the similarity transformation (7.4.5) $F_{a,o}$ must be stable as well.

which means the SDARE (7.4.1) has a solution. This concludes the proof of part (a).

The proofs of the remaining parts of the Theorem are much shorter. To prove part (b) if we assume that P is semistabilizing then, since the zeros of $M(z)$ and $N(z)$ are the eigenvalues of $F - K_pH$ and $F - K_qH$, it is straightforward to see that $M(z)$ and $N(z)$ are semistable. Therefore the main effort is to show the other direction. Thus, let us assume that a proper semi-canonical (semistabilizing) factorization of $\Sigma(z)$ exists. Now note that

$$T_a^{-1}(F - K_pH)T_a = \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} - \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} H_a & 0 & H_{a,c} & 0 \end{bmatrix}$$

$$\begin{bmatrix} F_a - K_aH_a & 0 & F_{a,13} - K_aH_{a,c} & 0 \\ F_{a,21} - X_aH_a & F_{a,o} & F_{a,23} - X_aH_{a,c} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix}$$

Thus the eigenvalues of $F - K_pH$ are the union of the eigenvalues of the matrices

$$\begin{bmatrix} F_a - K_aH_a & 0 \\ F_{a,21} - X_aH_a & F_{a,o} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F_{a,c} & 0 \\ 0 & F_{a,co} \end{bmatrix},$$

or, similarly, the matrices

$$F_a - K_aH_a \;\; , \;\; F_{a,o} \;\; , \;\; F_{a,c} \;\; \text{and} \;\; F_{a,co}.$$

Since the first of the above matrices is semi-stable (by the assumption that a semi-stable factorization exists) and the remaining three are stable, we conclude that $F - K_pH$ is semi-stable. A similar reasoning shows that $F - K_qH$ is semi-stable, thus ending our proof of part (b).

We now continue with the proof of part (c). First, suppose that the SDARE has a Hermitian solution, $P = P^*$. Then we can write the Popov function as

$$\Sigma(z) = \left[H(zI - F)^{-1}K_p + I\right] R_e \left[H(z^{-*}I - F)^{-1}K_p + I\right]^*, \qquad (7.4.12)$$

where $R_e = R + HPH^*$ is Hermitian. This readily shows the desired proper Hermitian factorization. [Note, moreover, that for all $|z| = 1$ that are neither poles nor zeros of $H(zI - F)^{-1}K_p + I$, the above relation shows that $\Sigma(z)$ is congruent to R_e. Therefore $\Sigma(z)$ will have constant inertia (equal to the inertia of R_e) a.e. on the unit circle, $|z| = 1$.]

In the other direction, suppose that we can write,

$$\Sigma(z) = M(z)JM^*(z^{-*}), \tag{7.4.13}$$

where $M(z)$ is a $p \times q$ proper rational transfer matrix with poles of magnitude less than one and where J is a $q \times q$ signature matrix. Repeating the arguments of the proof of part (a) to the above factorization shows that the SDARE will have a Hermitian solution. This ends the proof of part (c).

The proof of part (d) readily follows from parts (b) and (c).

Finally, to prove part (d), we note that (7.4.12) (in fact, its non-Hermitian counterpart) shows that $\Sigma(z)$ is nonsingular a.e. on the unit circle, $|z| = 1$, if, and only if, the matrix R_e is nonsingular. If this is the case, then R_e can be inverted in the SDARE, and we can rewrite the SDARE as the DARE

$$P = FPF^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^* + GQG^*.$$

∎

We have thus established a result on the existence, and properties of, solutions to the discrete-time algebraic Riccati equation (DARE) and the closely associated system of discrete-time algebraic Riccati equations (SDARE). The existence of these solutions, and their various properties, turns out to depend on the existence of certain proper factorizations, and their various properties, of the Popov function from which these equations are derived. Since we normally are interested in solving the DARE, or SDARE, in order to check whether the Popov function $\Sigma(z)$ has certain properties and admits certain factorizations (such as being positive, or having the canonical factorizations required of H^∞ estimation and control), it will be useful to have a separate method for solving these equations. One such method (based on the so-called Hamiltonian matrix) will be briefly studied in Sec. 7.6. However, before doing so, it will be useful to study the consequence of our results for some special cases of interest.

7.5 Special Cases

In this section we shall briefly review some special cases of the general DARE and SDARE that have been studied so far. These special cases are motivated by applications in linear quadratic control, linear least-squares (Wiener) filtering, and H^∞ control and filtering (see Secs. 1.3 and 1.6 and Secs. 1.4 and 1.7).

7.5.1 The Case of $R > 0$ and $Q - SR^{-1}S^* > 0$

We first remark that the case we are considering corresponds to the case where

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} > 0.$$

We should also note that there is no loss of generality in considering $Q - SR^{-1}S^* > 0$ instead of $Q - SR^{-1}S^* \geq 0$, since we can always go from the latter condition to the former one by a simple redefinition of the matrix G.

The so-called "positive" case for the DARE, that we are considering here, arises in linear quadratic control and linear least-squares filtering (see *e.g.*, Theorems 1.3.30 and 1.6.3) and is very well studied. Therefore, the results given below are well known (see *e.g.*, [AM79, Kuc74] and the references therein). However, for completeness, we will provide brief proofs using the approach presented so far.

Theorem 7.5.1 (The Positive DARE) *Consider the DARE*

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*, \qquad (7.5.1)$$

where $\{F, H\}$ *is detectable and* $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} > 0$. *Then we have the following results.*

(a) The DARE always has a Hermitian semi-stabilizing solution such that the matrix $F - K_pH$, *with* $K_p = (FPH^* + GS)(R + HPH^*)^{-1}$ *is semi-stable,* i.e., *has all its eigenvalues inside the closed unit disk.*

(b) *If, in addition,* $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ *is stabilizable, then the DARE always has a unique Hermitian and positive semi-definite stabilizing solution such that* $F - K_p H$ *is stable,* i.e., *has all its eigenvalues inside the open unit disk.*

(c) *If, in addition to the assumptions of part (b),* $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ *is controllable, then the unique stabilizing solution to the DARE is positive definite.*

Proof: To prove part (a), let us note that using the factorization

$$\begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} = \begin{bmatrix} I & GSR^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} GQG^* - GSR^{-1}S^*G^* & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & 0 \\ R^{-1}S^*G^* & I \end{bmatrix},$$

we may write the Popov function, $\Sigma(z)$, as

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} H(zI-F)^{-1}G & H(zI-F)^{-1}GSR^{-1} + I \end{bmatrix} \begin{bmatrix} Q - SR^{-1}S^* & 0 \\ 0 & R \end{bmatrix} \\ &\qquad \begin{bmatrix} G^*(z^{-1}I - F^*)^{-1}H^* \\ R^{-1}S^*G^*(z^{-1}I - F^*)^{-1}H^* + I \end{bmatrix} \\ &= H(zI-F)^{-1}G\left[Q - SR^{-1}S^*\right]G^*(z^{-1}I - F^*)^{-1}H^* + \\ &\qquad \left[H(zI-F)^{-1}GSR^{-1} + I\right] R \left[R^{-1}S^*G^*(z^{-1}I - F^*)^{-1}H^* + I\right]. \end{aligned}$$

The first term in the RHS of the above equation is a.e. positive semi-definite on the unit circle (since $Q - SR^{-1}S^* > 0$), whereas the second term is a.e. positive definite on the unit circle (since $R > 0$, and $H(zI-F)^{-1}GSR^{-1} + I$ is nonsingular for all $|z| = 1$ except, possibly, for its unit magnitude zeros). Therefore, $\Sigma(z)$ is positive definite a.e. on the unit circle, $|z| = 1$, so that, using Theorem 7.3.1, it admits a proper semi-canonical Hermitian factorization. Thus, from Theorem 7.2.1 part (d), the SDARE has a Hermitian semi-stabilizing solution, P. Moreover, using part (d) of the same Theorem, since $\Sigma(z)$ is nonsingular a.e. on the unit circle, then $R_e = R + HPH^*$ is nonsingular, and P is also a solution to the DARE (7.5.1). This concludes our proof of part (a).

The proofs for parts (b) and (c) are standard. We present them below.

For part (b) suppose that P is the Hermitian semi-stabilizing solution to the DARE, and that x is any left eigenvector of $F - K_pH$ with eigenvalue $|\lambda| = 1$, *i.e.*,

$$x(F - K_pH) = \lambda x.$$

Moreover, note that the DARE can be rewritten as

$$P = (F - K_pH)P(F - K_pH)^* + \begin{bmatrix} G & -K_p \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} G^* \\ -K_p^* \end{bmatrix}.$$

Pre and post-multiplying the above expression by x and x^*, respectively, we obtain

$$xPx^* = |\lambda|^2 xPx^* + x \begin{bmatrix} G & -K_p \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} G^* \\ -K_p^* \end{bmatrix} x^*,$$

from which we infer

$$x \begin{bmatrix} G & -K_p \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} G^* \\ -K_p^* \end{bmatrix} x^* = 0.$$

But this implies

$$x[G(Q - SR^{-1}S^*)G^* + (GSR^{-1} - K_p)R(GSR^{-1} - K_p)^*]x^* = 0,$$

from which we infer

$$xG(Q - SR^{-1}S^*)G^*x^* = 0 \quad \text{and} \quad xK_p = xGSR^{-1}. \tag{7.5.2}$$

We now see that

$$\lambda x = x(F - K_pH) = xF - xK_pH = xF - xGSR^{-1}H = x(F - GSR^{-1}H),$$

i.e. , x is also a left eigenvector of $F - GSR^{-1}H$ with eigenvalue λ. The first equality in (7.5.2) along with the above result can be rewritten as

$$xG(Q - SR^{-1}S^*) = 0 \quad \text{and} \quad x(F - GSR^{-1}H) = \lambda x.$$

This of course contradicts the stabilizability condition. We thus have

$$|\lambda| < 1,$$

meaning that $F - K_pH$ is stable. To show the uniqueness of the stabilizing solution, suppose that we have two solutions P_1 and P_2, with corresponding gain vectors $K_{p,1}$ and $K_{p,2}$. Now some algebra shows that

$$P_2 - P_1 = (F - K_{p,1}H)(P_2 - P_1)(F - K_{p,2}H)^*.$$

Applying the above equality i times, we have

$$P_2 - P_1 = (F - K_{p,1}H)^i(P_2 - P_1)(F - K_{p,2}H)^{i*}.$$

Now since the matrices $F - K_{p,1}H$ and $F - K_{p,2}H$ are both stable, as $i \to \infty$, the above equation becomes

$$P_2 - P_1 = 0$$

showing that the stabilizing solution is unique.

To prove part (c) we begin by writing the DARE as

$$P = (F - K_pH)P(F - K_pH)^* + G(Q - SR^{-1}S^*)G^* + (-K_p + GSR^{-1})R(-K_p + GSR^{-1})^*.$$

Since $F - K_pH$ is stable, using Lyapunov theory, P will be positive definite if the pair

$$\left\{F - K_pH, \left[\begin{array}{cc} GQ - GSR^{-1}S^* & -K_p + GSR^{-1} \end{array}\right]\right\}$$

is controllable. But since

$$\left[\begin{array}{ccc} F - GSR^{-1}H & GQ - GSR^{-1}S^* & -K_p + GSR^{-1} \end{array}\right] = \left[\begin{array}{ccc} F - K_pH & GQ - GSR^{-1}S^* - K_p + GSR^{-1} \end{array}\right] X$$

where

$$X = \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & I & 0 \\ -H & 0 & I \end{array}\right]$$

is a nonsingular matrix, the controllability of $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ will also yield a positive definite P. This establishes part (c).

∎

7.5.2 The Case of Nonsingular $F - GSR^{-1}H$

In the previous section we saw that when $R > 0$ and $Q - SR^{-1}S^* > 0$ then each solution to the SDARE was a solution of the DARE as well. Finding conditions under which R_e is nonsingular, so that solutions of the SDARE are solutions of the DARE as well, is, in the general case, quite difficult (see Ex. 4). However, it is possible to give a simple sufficient condition for this to be so.

Theorem 7.5.2 (Nonsingular $F - GSR^{-1}H$) *Suppose that $\{F, H\}$ is detectable and that R and $F - GSR^{-1}H$ are nonsingular. Then any solution to the SDARE (7.4.1) is also a solution to the DARE*

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

Proof: Let P be the solution to the SDARE

$$\begin{cases} P &= FPF^* + GQG^* - K_p R_e K_q^* \\ K_p R_e &= FPH^* + GS \quad , \quad K_q R_e^* = FP^*H^* + GS \\ R_e &= R + HPH^* \end{cases} .$$

Suppose now that R_e is singular. Then there exists an x such that

$$R_e x = 0.$$

Therefore if we post-multiply the second set of equations in the SDARE by x we obtain

$$(FPH^* + GS)x = 0.$$

But

$$FPH^* + GS = (F - GSR^{-1}H)PH^* + GSR^{-1}HPH^* + GS = (F - GSR^{-1}H)PH^* + GSR^{-1}R_e,$$

so that

$$(F - GSR^{-1}H)PH^* x = 0.$$

But since $F - GSR^{-1}H$ is nonsingular, this implies PH^*x. We can also conclude that $HPH^*x = 0$, so that

$$HPH^*x = (R_e - R)x = -Rx = 0,$$

which is a contradiction since R is assumed invertible. Therefore R_e must be nonsingular, and P is a solution to the DARE.

■

7.5.3 The Case of Positive $Q - SR^{-1}S^*$

Here we assume that $S = 0$, so that the DARE becomes

$$P = FPF^* + GQG^* - FPH^*(R + HPH^*)^{-1}HPF^*. \tag{7.5.3}$$

There is, of course, no loss of generality in making this assumption since the general DARE

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS),$$

can always be transformed to one with $S = 0$, by rewriting it as

$$\begin{aligned} P &= (F - GSR^{-1}H)P(F - GSR^{-1}H)^* + G(Q - SR^{-1}S^*)G^* - \\ &\quad (F - GSR^{-1}H)PH^*(R + HPH^*)^{-1}HP(F - GSR^{-1}H)^*, \end{aligned}$$

and redefining the matrices as

$$F \to F - GSR^{-1}H \quad \text{and} \quad Q \to Q - SR^{-1}S^*.$$

We therefore will be studying (7.5.3) in the case where $Q > 0$. Moreover, we can always assume that the (possibly) indefinite matrix R has the form

$$R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix}. \tag{7.5.4}$$

The reason being that the DARE can be rewritten as

$$P = F(I + PH^*R^{-1}H)^{-1}PF^* + GQG^*,$$

so that what really enters the DARE is the matrix $H^*R^{-1}H$ and a simple redefinition of H will allow R to have the form described in (7.5.4). We shall also partition H according to the partitioning of R and write

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

We should also remark that the DARE with $Q > 0$ and R indefinite arises in H^∞ filtering and control where it plays a prominent role [GD88, DGKF89, KN91, GL95]. [See also Theorems 1.4.2 and 1.7.3.] The following result on the DARE is known in the H^∞ literature, but is rarely stated in the form given below.

Theorem 7.5.3 (Positive Q) *Consider the DARE*

$$P = FPF^* + GQG^* - FPH^*(R + HPH^*)^{-1}HPF^*,$$

where

$$R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},$$

$\{F, H_1\}$ is detectable, $\{F, GQ^{1/2}\}$ is stabilizable and $Q > 0$. Suppose, moreover, that the DARE has a Hermitian stabilizing solution, P. Then we have the following results.

(a) $P \geq 0$.

(b) If, in addition, R and $R_e = R + HPH^*$ have the same inertia, then

(i) $F - K_1H$ is stable, where

$$K_1 = FPH_1^*(I_{p_1} + H_1PH_1^*)^{-1}.$$

(ii) The transfer matrix

$$M(z) = \begin{bmatrix} M_1(z) & M_2(z) \end{bmatrix},$$

where

$$M_1(z) = H_2(I + PH_1^*H_1)^{-1}(zI - F + K_1H_1)^{-1}GQ^{1/2}$$

and

$$M_2(z) = H_2PH_1^*(I_{p_1}+H_1PH_1^*)^{-1}+H_2(I+PH_1^*H_1)^{-1}(zI-F+K_1H_1)^{-1}K_1$$

is strictly contractive on the unit circle, i.e.,

$$M(z)M^*(z^{-*}) < I_{p_2}, \qquad \forall |z| = 1.$$

Remarks:

(a) Theorem 7.5.3 can be used to check whether the Popov function

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

admits the factorization required for the existence of suboptimal H^∞ filters. From Theorem 1.4.2, the condition for the existence of an H^∞ filter of level $\gamma = 1$ is that the Popov function admit the following canonical factorization,

$$\Sigma(z) = \begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix} \begin{bmatrix} L_{11}^*(z^{-*}) & L_{12}^*(z^{-*}) \\ L_{21}^*(z^{-*}) & L_{22}^*(z^{-*}) \end{bmatrix}, \qquad (7.5.5)$$

with $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ and $L_{11}(z)$ minimum phase and proper, and $L_{12}(z)$ stable and strictly proper. Now the existence of a canonical Hermitian factorization is equivalent to the existence of a Hermitian stabilizing solution to the DARE. That the factorization has the appropriate inertia is equivalent to R and R_e having the same inertia, and the strictly proper property of $L_{12}(z)$ and inverse stability of $L_{11}(z)$ can be checked by performing the block LDU decomposition of R_e and checking whether $F - K_1H$ is stable, where $K_1 = FPH_1^*(I_{p_1}+H_1PH_1^*)^{-1}$. Thus we have the following result.

"An H^∞ filter of level $\gamma = 1$ exists if, and only if, there exists a Hermitian stabilizing solution to the DARE such that the matrices $R = I_{p_1} \oplus (-I_{p_2})$ and $R_e = R + HPH^*$ have the same inertia." But this is precisely the what is stated in the second part of Theorem 1.4.2.

(b) The transfer matrix $M(z)$ of Theorem 7.5.3 can be seen to be equal to $T_{K_{cen}}(z)$, the transfer operator that maps the disturbances to estimation errors in the H^∞ filtering problem of level $\gamma = 1$, when the central filter is used. Statement (b), part (ii), of Theorem 7.5.3 states the (obvious) fact that when a solution to the H^∞ filtering problem exists $T_{K_{cen}}(z)$ is strictly contractive.

Proof of Theorem 7.5.3: Let us first prove part (a) by establishing $P \geq 0$. To this end, consider, P_γ, the stabilizing solution to the DARE[11]

$$P_\gamma = FP_\gamma F^* + GQG^* - FP_\gamma H^*(R_\gamma + HP_\gamma H^*)^{-1}HP_\gamma F^*, \tag{7.5.6}$$

where

$$R_\gamma = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -\gamma I_{p_2} \end{bmatrix}, \quad \gamma \geq 1.$$

This DARE may be written as

$$P_\gamma = F(I + P_\gamma H_1^* H_1 - \gamma^{-1} P_\gamma H_2^* H_2)^{-1} P_\gamma F^* + GQG^*. \tag{7.5.7}$$

We can use the above expression to differentiate P_γ with respect to γ and obtain $\dot{P}_\gamma = dP_\gamma/d\gamma$. Using the identity,

$$\frac{dA^{-1}}{d\gamma} = -A^{-1}\frac{dA}{d\gamma}A^{-1},$$

and after some algebra, we may write

$$\dot{P}_\gamma = F(I+P_\gamma H_1^* H_1 - \gamma^{-1}P_\gamma H_2^* H_2)^{-1}(P_\gamma - \gamma^{-2}P_\gamma H_2^* H_2 P_\gamma)(I+P_\gamma H_1^* H_1 - \gamma^{-1}P_\gamma H_2^* H_2)^{-*}F^*. \tag{7.5.8}$$

But note that since P_γ is stabilizing, the matrix

$$F(I + P_\gamma H_1^* H_1 - \gamma^{-1} P_\gamma H_2^* H_2)^{-1} = F(I + P_\gamma H^* R_\gamma^{-1} H^*)^{-1} = F - K_{p,\gamma} H,$$

is stable. Thus (7.5.8) becomes

$$\dot{P}_\gamma = (F - K_{p,\gamma}H)P_\gamma(F - K_{p,\gamma}H)^* - \gamma^{-2}(F - K_{p,\gamma}H)P_\gamma H_2^* H_2 P_\gamma (F - K_{p,\gamma}H)^*.$$

[11] The given DARE is simply the DARE corresponding to the H^∞ filtering problem with level $\gamma \geq 1$. Since we have assumed that the level $\gamma = 1$ H^∞ filtering problem has a solution, the same will be true for all $\gamma \geq 1$. Therefore, for $\gamma \geq 1$ the given DARE will have a Hermitian stabilizing solution, as stated.

Since the second term in the above equation is negative semi-definite and since $F - K_p H$ is stable, using Lyapunov theory, we have

$$\dot{P}_\gamma \leq 0.$$

Therefore, for all γ such that P_γ is stabilizing, the P_γ form a nonincreasing function of Hermitian matrices. In other words, if P_{γ_1} and P_{γ_2} are two stabilizing solutions to the DARE (7.5.6), then

$$\gamma_1 \leq \gamma_2 \quad \text{implies} \quad P_{\gamma_1} \geq P_{\gamma_2}.$$

Now if we let $\gamma \to \infty$, then (7.5.7) becomes

$$P_\infty = F(I+P_\infty H_1^* H_1)^{-1} P_\infty F^* + GQG^* = FP_\infty F^* + GQG^* - FP_\infty H_1^*(I_{p_1} + H_1 P_\infty H_1^*)^{-1} H_1 P_\infty F^*.$$

But since $Q > 0$, this is an example of the positive DARE studied in Sec. 7.5.1. Therefore, since $\{F, H_1\}$ is detectable and $\{F, GQ^{1/2}\}$ is stabilizable, using Theorem 7.5.1 part (b), a stabilizing solution to this DARE exists and is such that $P_\infty \geq 0$. Now from the hypothesis of the Theorem we know that the DARE (7.5.6) has a stabilizing solution for $\gamma = 1$. As $\gamma = 1 < \infty$, using the (above established) monotonicity property of the stabilizing solutions to (7.5.6), we have

$$P = P_1 \geq P_\infty \geq 0,$$

which proves part (a).

To proceed with the proof of part (b), we now assume that the matrices

$$R_e = R + HPH^* = \begin{bmatrix} I_{p_1} + H_1 P H_1^* & H_1 P H_2^* \\ H_2 P H_1^* & -I_{p_2} + H_2 P H_2^* \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix},$$

have the same inertia. Since $P \geq 0$, we have $I_{p_1} + H_1 P H_1^* > 0$, and therefore the above inertia condition implies that the Schur complement of the $(1,1)$ block entry of R_e must be negative definite, *i.e.*,

$$-I_{p_2} + H_2 P H_2^* - H_2 P H_1^*(I_{p_1} + H_1 P H_1^*)^{-1} H_1 P H_2^* = -I_{p_2} + H_2(I + PH_1^* H_1)^{-1} P H_2^* < 0.$$

Now we can write the DARE as

$$\begin{aligned}
P &= F(I+PH_1^*H_1-PH_2^*H_2)^{-1}PF^*+GQG^* \\
&= F(I+PH_1^*H_1)^{-1}PF^*+GQG^* \\
&\quad -F(I+PH_1^*H_1)^{-1}PH_2^*\left[H_2(I+PH_1^*H_1)^{-1}PH_2^*-I_{p_2}\right]^{-1}H_2(I+PH_1^*H_1)^{-1}PF^* \\
&= F(I+PH_1^*H_1)^{-1}PF^*+GQG^* \\
&\quad +F(I+PH_1^*H_1)^{-1}PH_2^*\underbrace{\left[I_{p_2}-H_2(I+PH_1^*H_1)^{-1}PH_2^*\right]^{-1}}_{=M>0}H_2P(I+H_1^*H_1P)^{-1}F^* \\
&= (F-K_1H)P(F-K_1H)^*+GQG^*+K_1K_1^*+\underbrace{F(I+PH_1^*H_1)^{-1}PH_2^*MH_2P(I+H_1^*H_1P)^{-1}F^*}_{=\bar{Q}\geq 0},
\end{aligned}$$

where in the second step we have expanded $(I+PH_1^*H_1-PH_2^*H_2)^{-1}$ using the matrix inversion lemma, and where in the fourth step we have used the expression $K_1 = FPH_1^*(I_{p_1}+H_1PH_1^*)^{-1}$. Now since $P\geq 0$, $\{F,G^{1/2}\}$ is stabilizable, and $GQG^*+K_1K_1^*+\bar{Q}\geq$, we can use an argument similar to the argument of Theorem 7.5.1 part (b) to show that $F-K_1H$ is stable. This proves part (b-i) of this Theorem.

To prove part (b-ii) we proceed as follows. First, note that using the stabilizing solution to the DARE we can factorize the Popov function as

$$\Sigma(z)=\begin{bmatrix}\mathcal{L}_{11}(z) & \mathcal{L}_{12}(z)\\ \mathcal{L}_{21}(z) & \mathcal{L}_{22}(z)\end{bmatrix}\begin{bmatrix}R_{e,1}I_{p_1} & 0\\ 0 & -R_{e,2}I_{p_2}\end{bmatrix}\begin{bmatrix}\mathcal{L}_{11}(z^{-*}) & \mathcal{L}_{21}(z^{-*})\\ \mathcal{L}_{21}(z^{-*}) & \mathcal{L}_{22}(z^{-*})\end{bmatrix}^*, \quad (7.5.9)$$

where both $R_{e,1}$ and $R_{e,2}$ are positive definite. (Such a factorization is called a J-spectral factorization.) Indeed, this factorization is achieved via

$$\Sigma(z)=\left\{\begin{bmatrix}H_1\\ H_2\end{bmatrix}(zI-F)^{-1}K_p+\begin{bmatrix}I_{p_1} & 0\\ 0 & I_{p_2}\end{bmatrix}\right\}\begin{bmatrix}I_{p_1}+H_1PH_1^* & H_1PH_2^*\\ H_2PH_1^* & -I_{p_2}+H_2PH_2^*\end{bmatrix}\{\}^*,$$

and the block triangular factorization of the central matrix

$$\begin{bmatrix}I_{p_1}+H_1PH_1^* & H_1PH_2^*\\ H_2PH_1^* & -I_{p_2}+H_2PH_2^*\end{bmatrix}=$$

$$\begin{bmatrix}I_{p_1} & 0\\ H_2PH_1^*(I_{p_1}+H_1PH_1^*)^{-1} & I_{p_2}\end{bmatrix}\begin{bmatrix}I_{p_1}+H_1PH_1^* & 0\\ 0 & -R_{e,2}\end{bmatrix}\begin{bmatrix}I_{p_1} & (I_{p_1}+H_1PH_1^*)^{-1}H_1PH_2^*\\ 0 & I_{p_2}\end{bmatrix},$$

where $R_{e,2}=I_{p_2}-H_2(I+PH_1^*H_1)^{-1}PH_2^*>0$. Thus it is readily seen that

$$\begin{bmatrix}\mathcal{L}_{11}(z) & \mathcal{L}_{12}(z)\\ \mathcal{L}_{21}(z) & \mathcal{L}_{22}(z)\end{bmatrix}=\left\{\begin{bmatrix}H_1\\ H_2\end{bmatrix}(zI-F)^{-1}K_p+\begin{bmatrix}I_{p_1} & 0\\ 0 & I_{p_2}\end{bmatrix}\right\}\begin{bmatrix}I_{p_1} & 0\\ H_2PH_1^*(I_{p_1}+H_1PH_1^*)^{-1} & I_{p_2}\end{bmatrix}.$$

Some algebra now shows that

$$\mathcal{L}_{11}(z) = H_1(zI - F)^{-1}K_1 + I_{p_1}, \tag{7.5.10}$$

and

$$\mathcal{L}_{21}(z) = H_2(zI - F)^{-1}K_1 + H_2PH_1^*(I_{p_1} + H_1PH_1^*)^{-1}. \tag{7.5.11}$$

We may also remark that since $F - K_1H$ is stable, the transfer matrix $\mathcal{L}_{11}(z)$ has a stable inverse. Now using the factorization (7.5.9) it is straightforward to see

$$\begin{aligned} &\begin{bmatrix} -\mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z) & I_{p_2} \end{bmatrix} \Sigma(z) \begin{bmatrix} \mathcal{L}_{11}^{-*}(z^{-*})\mathcal{L}_{21}^*(z^{-*}) \\ I_{p_2} \end{bmatrix} = \\ &-\left(\mathcal{L}_{22}(z) - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{L}_{12}(z)\right) R_{e,2} \left(\mathcal{L}_{22}(z^{-*}) - \mathcal{L}_{21}(z^{-*})\mathcal{L}_{11}^{-1}(z^{-*})\mathcal{L}_{12}(z^{-*})\right)^* < 0, \end{aligned} \tag{7.5.12}$$

for all $|z| = 1$. Note that the inequality is strict since $\mathcal{L}_{22} - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{L}_{12}(z)$ has no unit circles zeros (in turn because $\begin{bmatrix} \mathcal{L}_{11}(z) & \mathcal{L}_{12}(z) \\ \mathcal{L}_{21}(z) & \mathcal{L}_{22}(z) \end{bmatrix}$ has no unit circle zeros – since P is stabilizing – and $\mathcal{L}_{11}(z)$ has a stable inverse). On the other hand, if we define

$$\mathcal{H}_1(z) = H_1(zI - F)^{-1}GQ^{1/2} \quad \text{and} \quad \mathcal{H}_2(z) = H_2(zI - F)^{-1}GQ^{1/2}, \tag{7.5.13}$$

we may write

$$\Sigma(z) = \begin{bmatrix} I_{p_1} + \mathcal{H}_1(z)\mathcal{H}_1^*(z^{-*}) & \mathcal{H}_1(z)\mathcal{H}_2^*(z^{-*}) \\ \mathcal{H}_2(z)\mathcal{H}_1^*(z^{-*}) & -I_{p_2} + \mathcal{H}_2(z)\mathcal{H}_2^*(z^{-*}) \end{bmatrix}.$$

With the above expression for the Popov function we readily see that

$$\begin{aligned} &\begin{bmatrix} -\mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z) & I_{p_2} \end{bmatrix} \Sigma(z) \begin{bmatrix} \mathcal{L}_{11}^{-*}(z^{-*})\mathcal{L}_{21}^*(z^{-*}) \\ I_{p_2} \end{bmatrix} = \\ &\left(\mathcal{H}_2(z) - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{H}_1(z)\right)\left(\mathcal{H}_2(z^{-*}) - \mathcal{L}_{21}(z^{-*})\mathcal{L}_{11}^{-1}(z^{-*})\mathcal{H}_1(z^{-*})\right)^* \\ &\quad + \left(\mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\right)\left(\mathcal{L}_{21}(z^{-*})\mathcal{L}_{11}^{-1}(z^{-*})\right)^* - I_{p_1}. \end{aligned}$$

Coupling the above equation with the strict inequality of (7.5.12) we obtain

$$\begin{aligned} &\left(\mathcal{H}_2(z) - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{H}_1(z)\right)\left(\mathcal{H}_2(z^{-*}) - \mathcal{L}_{21}(z^{-*})\mathcal{L}_{11}^{-1}(z^{-*})\mathcal{H}_1(z^{-*})\right)^* \\ &\quad + \left(\mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\right)\left(\mathcal{L}_{21}(z^{-*})\mathcal{L}_{11}^{-1}(z^{-*})\right)^* < I_{p_1}, \end{aligned}$$

for all $|z| = 1$. Therefore the transfer matrix

$$\left[\begin{array}{cc} \mathcal{H}_2(z) - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{H}_1(z) & \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z) \end{array} \right],$$

is strictly contractive on the unit circle. Using the expressions for $\mathcal{L}_{11}(z)$ and $\mathcal{L}_{21}(z)$ (see (7.5.10-7.5.11)) and those for $\mathcal{H}_1(z)$ and $\mathcal{H}_2(z)$ (see (7.5.13)), it is not too difficult to show that

$$\mathcal{H}_2(z) - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{H}_1(z) = H_2(I + PH_1^*H_1)^{-1}(zI - F + K_1H_1)^{-1}GQ^{1/2},$$

and

$$\mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z) = H_2PH_1^*(I_{p_1} + H_1PH_1^*)^{-1} + H_2(I + PH_1^*H_1)^{-1}(zI - F + K_1H_1)^{-1}K_1,$$

thus proving part (b-ii).

∎

7.6 The Hamiltonian Matrix

In the previous sections we developed general conditions for the existence of solutions to the DARE, and the closely related SDARE, in terms of the existence of certain proper factorizations of the associated Popov function. We then characterized the properties of these solutions in terms of the properties of the aforementioned factorizations. However, in most applications, such as in H^∞ problems, one is interested in establishing the properties of the Popov function itself, *e.g.*, whether a canonical factorization of a certain type exists or not. Therefore it would be very useful (and indeed essential) to have an independent means of checking whether a solution to the DARE exists, and if so, of actually computing this solution. If this is possible, then one can then infer the properties of the Popov function from the properties of the (computed) Riccati solution.

There now exists a vast (and growing) literature on how to obtain solutions to algebraic Riccati equations, albeit mostly in the positive case (see the survey paper [Lau91] and the references therein). Although it is beyond the scope of this chapter to study (or even mention) all of the various methods for computing solutions to the DARE, in this section we shall investigate one possible method for computing the semi-stabilizing solution to the DARE. We shall essentially show that the wellknown method of invariant subspaces of the Hamiltonian matrix [Pot66, Mac63, Lau79] extends to the case of DARE's with indefinite Q and R.

We also remark that in what follows, without loss of generality, we shall assume that $S = 0$.

7.6.1 The Case of Nonsingular F

We begin by assuming that F is nonsingular. This simplifies some of the arguments. More importantly, from Theorem 7.5.2, it implies that, whenever a solution to the SDARE exists, a solution to the DARE also exists. Moreover, it also implies that the matrix

$$F - K_p H = F - FPH^*(R + HPH^*)^{-1} = F(I + PH^*R^{-1}H)^{-1},$$

is always invertible. Studying the general case of a (possibly) singular F is more complicated. We mention what may amount to a promising route for this problem in the next section.

We continue by attempting to write the DARE as a quadratic form in P. To this end, we first note that the DARE may be rewritten as

$$-P + F(I + PH^*R^{-1}H)^{-1}PF^* + GQG^* = 0,$$

or, upon premultiplying by $(I + PH^*R^{-1}H)F^{-1}$, as

$$(I + PH^*R^{-1}H)F^{-1}(-P + GQG^*) + PF^* = 0.$$

Gathering terms yields

$$\begin{bmatrix} I & -P \end{bmatrix} \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix} \begin{bmatrix} P \\ I \end{bmatrix} = 0, \qquad (7.6.1)$$

which is the desired quadratic form. The center matrix appearing in the above expression is referred to as the *Hamiltonian* matrix and is denoted by

$$M = \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix}. \tag{7.6.2}$$

It is convenient to expand the block row and column vectors appearing in (7.6.1) into upper triangular matrices, to obtain

$$\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix} \begin{bmatrix} I & P \\ 0 & I \end{bmatrix}.$$

Due to (7.6.1), the $(1,2)$ block entry in the above resulting product is zero. Some simple algebra shows that the remaining block entries are

$$\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} = \begin{bmatrix} F_p^{-1} & 0 \\ -H^*R^{-1}HF^{-1} & F_q^* \end{bmatrix} \tag{7.6.3}$$

where we have defined

$$F_p = F - K_pH \quad \text{and} \quad F_q = F - K_qH. \tag{7.6.4}$$

The identity (7.6.3) has special significance since it leads to the following result on the eigenvalue structure of the Hamiltonian matrix M.

Lemma 7.6.1 (Eigenvalues of the Hamiltonian Matrix) *Consider the Hamiltonian matrix*

$$M = \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix}.$$

Then λ is an eigenvalue of M if, and only if, λ^{-*} is an eigenvalue of M. Moreover, M can be decomposed as

$$T^{-1}MT = \begin{bmatrix} \Lambda_1 & \times \\ 0 & \Lambda_2^{-*} \end{bmatrix}, \tag{7.6.5}$$

where Λ_1 and Λ_2 have eigenvalues of magnitude less than or equal to one, and '$\times$' denotes irrelevant entries. Finally, if λ is an eigenvalue of Λ_1, such that $|\lambda| < 1$, then λ is an eigenvalue of Λ_2 as well.

Proof: The results are a direct consequence of (7.6.3). Note that since

$$\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix}^{-1},$$

the Hamiltonian M is similar to the block diagonal matrix

$$\begin{bmatrix} F_p^{-1} & 0 \\ -H^*R^{-1}HF^{-1} & F_q^* \end{bmatrix}.$$

Therefore the set of eigenvalues of M is the union of the set of eigenvalues of F_p^{-1} and the set of eigenvalues of F_q^*. Now from the factorization of the Popov function,

$$\Sigma(z) = \left[H(zI - F)^{-1}K_p + I\right] R_e \left[H(z^{-*}I - F)^{-1}K_q + I\right]^*,$$

we see that the zeros of $M(z)$ are the eigenvalues of F_p and F_q^{-*}. Therefore, we conclude that the eigenvalues of M are simply the *inverse* of the zeros of the Popov function, $\Sigma(z)$. Since $\Sigma(z)$ is para-Hermitian, for every zero at σ there exists a zero at σ^{-*}. The same is therefore true of the eigenvalues of M. This establishes the first claim of the Lemma.

The decomposition (7.6.5) follows immediately from (7.6.3). We can then readily identify the Λ_1 with the eigenvalues of F_q^* and Λ_2 with the eigenvalues of F_p^*, which means that they both have eigenvalues of magnitude less than or equal to one (since F_p and F_q are semi-stable). Due to the above factorization of the para-Hermitian Popov function, $\Sigma(z)$, we see that F_p and F_q have the same eigenvalues of magnitude strictly less than one. The same is therefore true of Λ_1 and Λ_2, thus finishing the proof of this Lemma.

■

The above Lemma furnishes a method for computing solutions to the DARE (when they exist). To see why, let us suppose that P is a solution of the DARE and write (7.6.3) as

$$M\begin{bmatrix} I & P \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix}\begin{bmatrix} F_p^{-1} & 0 \\ -H^*R^{-1}HF^{-1} & F_q^* \end{bmatrix},$$

from which we may conclude

$$M \begin{bmatrix} P \\ I \end{bmatrix} = \begin{bmatrix} P \\ I \end{bmatrix} F_q^*. \tag{7.6.6}$$

Now from the proof of Lemma 7.6.1, we have identified the eigenvalue structure of F_q^* with Λ_1. Therefore there must exist a similarity transformation relating these two matrices, *i.e.*,

$$F_q^* = W\Lambda_1 W^{-1},$$

so that we may write (7.6.6) as

$$M \begin{bmatrix} PW \\ W \end{bmatrix} = \begin{bmatrix} PW \\ W \end{bmatrix} \Lambda_1.$$

If we now define $T_{21} = W$ and $T_{11} = PW$, then we may write

$$M \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} \Lambda_1. \tag{7.6.7}$$

Note that in the above expression the matrix T_{21} is invertible. This means that P, the solution to the DARE, may be found via

$$P = T_{11}T_{21}^{-1}. \tag{7.6.8}$$

This is an important result since it has allowed us to write the stabilizing solution of the DARE in terms of elements of the (generalized) eigenvalue decomposition of M. We formalize this result in the following Theorem.

Theorem 7.6.1 (The Invariant Subspace Method) *Consider the DARE*

$$P = FPF^* + GQG^* = FPH^*(R + HPH^*)^{-1}HPF^*,$$

and suppose that F is invertible and $\{F, H\}$ is detectable. Then we have the following results.

(a) *The DARE has a solution if, and only if, there exists a basis,* $\begin{bmatrix} T_{11}^* & T_{21}^* \end{bmatrix}^*$, *for some n-dimensional invariant subspace of the Hamiltonian matrix, M, such that T_{21} is invertible. In other words, if, and only if, there exist T_{11} and T_{21} given by*

$$\begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} \Lambda_1, \quad (7.6.9)$$

where Λ_1 is an $n \times n$ matrix that represents n (of the $2n$) eigenvalues of M, and T_{21} is invertible. In this case, the solution of the DARE is given by

$$P = T_{11}T_{21}^{-1}. \quad (7.6.10)$$

(b) *The DARE has a semistabilizing solution if, and only if, there exists a basis,* $\begin{bmatrix} T_{11}^* & T_{21}^* \end{bmatrix}^*$, *for some n-dimensional semistable invariant subspace of the Hamiltonian matrix, M, such that T_{21} is invertible. In other words, if, and only if, there exist T_{11} and T_{21} given by (7.6.9), where now Λ_1 is an $n \times n$ matrix that has all its eigenvalues inside the closed unit disk, and T_{21} is invertible. In this case, P is given by (7.6.10).*

(c) *The DARE has a stabilizing solution if, and only if, the Hamiltonian matrix, M, has no unit circle eigenvalues, and,* $\begin{bmatrix} T_{11}^* & T_{21}^* \end{bmatrix}^*$, *the n-dimensional stable invariant subspace of M is such that T_{21} is invertible. In this case, P is given by (7.6.10).*

Proof: The proof is quite straightforward. We have already shown that if a solution to the DARE exists then a basis for some invariant subspace of M with T_{21} invertible exists. On the other hand, it is easy to show that if a basis, with the aforementioned properties, for some invariant subspace of M exists, then $T_{11}T_{21}^{-1}$ is a solution to the DARE.

Finally, the claims of parts (b) and (c) follow from the fact that the eigenvalues of Λ_1 (associated with the eigenvalues of the invariant subspace under consideration) coincide with the eigenvalues of F_q^*. Moreover, M can only have a stable invariant subspace if it has no unit circle eigenvalues. (Otherwise we only have semistable invariant subspaces.)

∎

The connection between an apparently nonlinear matrix Riccati equation of order n and a linear eigenvalue problem of order $2n$ is classical and dates back at least to Von Escherich in 1898 [Esc98]. The above "eigenvector" solution was popularized in the control literature by MacFarlane in 1963 [Mac63] and Potter in 1966 [Pot66]. Schur methods for the solution of this eigenvalue problem have been proposed by Laub in 1978 [Lau79], and have resulted in reliable and efficient numerical solutions to algebraic Riccati equations in the positive case.

The above result essentially claims that solutions to the DARE can also be found using the invariant subspace method even in the indefinite case. However, the study of reliable numerical methods for obtaining this solution is beyond the scope of this chapter and is worthy of further scrutiny.

7.6.2 The Case of Singular F

We note that when F is invertible, the Hamiltonian matrix M may be written as

$$M = \begin{bmatrix} I & 0 \\ -H^*R^{-1}H & F^* \end{bmatrix} \begin{bmatrix} F & GQG^* \\ 0 & I \end{bmatrix}^{-1}.$$

This therefore suggests that in the case of singular F, instead of computing the eigenvalues and eigenvectors of M, we should compute the generalized eigenvalues and eigenvectors of the pair

$$\left\{ \begin{bmatrix} I & 0 \\ -H^*R^{-1}H & F^* \end{bmatrix}, \begin{bmatrix} F & GQG^* \\ 0 & I \end{bmatrix} \right\}.$$

In the positive case, this approach has been used for computing solutions to the DARE when F is singular [TPS80]. We believe such an approach should extend to the indefinite case although we shall not pursue it here.

7.7 Some Examples

In this section we shall consider some simple examples that will demonstrate the results obtained so far. Our first example concerns how to use solutions of the DARE to compute bounds for the H^∞ norm of a certain transfer function.

Example 1 (Computation of the H^∞ Norm) Consider the transfer function

$$A(z) = \frac{1}{z - 1/2}, \tag{7.7.1}$$

and suppose we would like to find constants γ_u and γ_l such that

$$\gamma_l^2 \leq |A(e^{j\omega})|^2 \leq \gamma_u^2, \quad \forall \omega \in [0, 2\pi]. \tag{7.7.2}$$

Note that the smallest possible γ_u (*i.e.*, one that makes the above inequality tight) is the H^∞ norm of $A(z)$. The expression (7.7.2) suggests that we introduce the Popov function

$$\Sigma(z) = \gamma^2 - A(z)A^*(z^{-*}). \tag{7.7.3}$$

The inertia of $\Sigma(z)$ on the unit circle, for different values of γ, will allow us to find γ_u and γ_l in (7.7.2). But let us first write $\Sigma(z)$ in the standard form

$$\Sigma(z) = \begin{bmatrix} (z-1/2)^{-1} & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & \gamma^2 \end{bmatrix} \begin{bmatrix} (z^{-1}-1/2)^{-1} \\ 1 \end{bmatrix}. \tag{7.7.4}$$

Comparing with the general Popov function (7.2.6), we see that we can identify

$$F = 1/2 \quad , \quad G = 1 \quad , \quad H = 1 \quad , \quad Q = -1 \quad , \quad S = 0 \quad , \quad R = \gamma^2.$$

Thus the DARE associated with $\Sigma(z)$ is given by

$$p = p/4 - 1 - \frac{p^2/4}{\gamma^2 + p},$$

or, after some rearrangement of terms,

$$4p^2 + (3\gamma^2 + 4)p + 4\gamma^2 = 0. \tag{7.7.5}$$

Note that, being a quadratic equation in p, the above DARE always has a solution (see Theorem 7.4.1). Now solutions to the DARE are given by

$$p_{1,2} = \frac{1}{8} \cdot \left(-3\gamma^2 - 4 \pm \sqrt{9\gamma^4 - 40\gamma^2 + 16} \right).$$

The DARE will therefore have real (corresponding to Hermitian) solutions if, and only if,

$$9\gamma^4 - 40\gamma^2 + 16 \geq 0,$$

or equivalently, if, and only if,

$$\gamma^2 \geq 4 \quad \text{or} \quad \gamma^2 \leq \frac{4}{9}.$$

Thus for the above values of γ, the Popov function, $\Sigma(z)$, admits Hermitian factorizations, and will have constant inertia on the unit circle. By Theorem 7.4.1, this constant inertia is given by the inertia of R_e. But

$$R_{e_{1,2}} = \gamma^2 + p_{1,2} = \frac{1}{8} \cdot \left[5\gamma^2 - 4 \pm \sqrt{(5\gamma^2 - 4)^2 - 16\gamma^4} \right].$$

Therefore for $\gamma^2 \geq 4$, we have $R_e > 0$, and for $\gamma^2 \leq \frac{4}{9}$, we have $R_e < 0$. Hence $\Sigma(z)$ is non-negative on the unit circle for all $\gamma^2 \geq 4$, and is non-positive on the unit circle for all $\gamma^2 \leq \frac{4}{9}$.[12] Thus, we conclude that

$$\gamma_u^2 = \sup_{\omega \in [0,2\pi]} |A(e^{j\omega})|^2 = 4 \quad \text{and} \quad \gamma_l^2 = \inf_{\omega \in [0,2\pi]} |A(e^{j\omega})|^2 = \frac{4}{9},$$

which is what we had set out to find.

We should also mention that for intermediate values of γ, *i.e.*, $\frac{4}{9} < \gamma^2 < 4$, the DARE does not have a real (Hermitian) solution, and that therefore the Popov function $\Sigma(z)$ does not have constant inertia on the unit circle, $|z| = 1$.

■

Example 2 (Properties of the DARE) We now continue with the DARE of Example 1 and establish some of its properties. First, we verify that it always has a semi-stabilizing solution. To this end, note that we may write

$$F_p = F - K_p H = F(I + PH^* R^{-1} H)^{-1} = \frac{FR}{R + HPH^*} = \frac{FR}{R_e}.$$

[12]Correspondingly, $\Sigma(z)$ is positive and negative on the unit circle for all $\gamma^2 > 4$ and $\gamma^2 < \frac{4}{9}$, respectively.

Therefore

$$F_{p_{1,2}} = \frac{\gamma^2/2}{\frac{1}{8}\cdot\left[5\gamma^2 - 4 \pm \sqrt{(5\gamma^2-4)^2 - 16\gamma^4}\right]} = \frac{4\gamma^2}{5\gamma^2 - 4 \pm \sqrt{(5\gamma^2-4)^2 - 16\gamma^4}}.$$

Now it is straightforward to see that

$$F_{p_1}F_{p_2} = 1.$$

Therefore at least one of F_{p_1} or F_{p_2} is semi-stable, meaning that at least one of p_1 or p_2 is semi-stabilizing. To be more specific, when $\gamma^2 > 4$, we have

$$F_{p_1} = \frac{4\gamma^2}{4\gamma^2 + (\gamma^2 - 4) + \sqrt{(5\gamma^2-4)^2 - 16\gamma^4}} < 1,$$

so that p_1 is the stabilizing solution. Likewise, when $\gamma^2 < \frac{4}{9}$, we have

$$F_{p_2} = \frac{4\gamma^2}{-4\gamma^2 - (4 - 9\gamma^2) - \sqrt{(5\gamma^2-4)^2 - 16\gamma^4}} > -1,$$

so that $-1 < F_{p_2} < 0$, and p_2 is the stabilizing solution. In either case, we have $K_p = K_q$ (where K_p corresponds to the stabilizing solution), so that the spectral factorization of $\Sigma(z)$ is given by

$$\Sigma(z) = \left[\frac{K_p}{z - 1/2} + 1\right] R_e \left[\frac{K_p}{z^{-1} - 1/2} + 1\right] = \frac{z - 1/2 + K_p}{z - 1/2} R_e \frac{z^{-1} - 1/2 + K_p}{z^{-1} - 1/2}.$$

In the case where $\frac{4}{9} < \gamma^2 < 4$, we have

$$F_{p_{1,2}} = \frac{4\gamma^2}{5\gamma^2 - 4 \pm j\sqrt{16\gamma^4 - (5\gamma^2-4)^2}}.$$

Therefore, it is easy to see that

$$|F_{p_{1,2}}| = 1,$$

so that *both* p_1 and p_2 are semi-stabilizing solutions. In this case, we have $F_{p_1} = F_{p_2}^*$, so that the spectral factorization of $\Sigma(z)$ is given by

$$\Sigma(z) = \frac{z - F_p}{z - 1/2} R_e \frac{z^{-1} - F_p}{z^{-1} - 1/2}, \qquad |F_p| = 1.$$

If we compare this factorization with the factorization $\Sigma(z) = M(z)N^*(z^{-*})$, given in Lemma 7.3.3, we can identify

$$M(z) = \frac{z - F_p}{z - 1/2} \quad \text{and} \quad N(z) = R_e^* \frac{z - F_p^*}{z - 1/2}.$$

Therefore the zero of $N(z)$ is the complex conjugate of the zero of $M(z)$.

■

Example 3 (The Hamiltonian Matrix) We once more return to the DARE of Example 1, and now study its associated Hamiltonian matrix

$$M = \begin{bmatrix} 2 & 2 \\ -2\gamma^{-2} & 1/2 - 2\gamma^{-2} \end{bmatrix}. \tag{7.7.6}$$

It is easy to see that the characteristic equation corresponding to M is

$$\lambda^2 - (5/2 - 2\gamma^{-2})\lambda + 1 = 0,$$

whose solutions are

$$\lambda_{1,2} = \frac{5 - 4\gamma^{-2} \pm \sqrt{(5 - 4\gamma^{-2})^2 - 16}}{4}.$$

Suppose, for example, that $\gamma^2 < \frac{4}{9}$. In this case the stable eigenvalue of M is given by

$$-1 < \lambda_1 = \frac{5 - 4\gamma^{-2} + \sqrt{(5 - 4\gamma^{-2})^2 - 16}}{4} < 0.$$

A basis for the stable invariant subspace of M can be found by solving the equations

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix},$$

which yields

$$x = 1 \quad \text{and} \quad y = \frac{-3 - 4\gamma^{-2} + \sqrt{(5 - 4\gamma^{-2})^2 - 16}}{8}.$$

Thus, the stabilizing solution to the DARE is given by

$$p = xy^{-1} = \frac{-3\gamma^2 - 4 - \sqrt{(5\gamma^2 - 4)^2 - 16\gamma^2}}{8},$$

which corresponds to the solution obtained in Example 1.

As an example of the case where $\frac{4}{9} \leq \gamma^2 \leq 4$, let us consider $\gamma = 1$. In this case we have,

$$\lambda_{1,2} = \frac{1 \pm j\sqrt{15}}{4},$$

and therefore a basis for a semi-stable invariant subspace of M can be found by solving the equations

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1 + j\sqrt{15}}{4} \begin{bmatrix} x \\ y \end{bmatrix},$$

which yields

$$x = 1 \quad \text{and} \quad y = \frac{-7 + j\sqrt{15}}{8},$$

and from which we infer that a semi-stabilizing solution to the DARE is given by

$$p = xy^{-1} = \frac{-7 - j\sqrt{15}}{8}.$$

Finally, another case of interest is when $\gamma^2 = 4$, since in this case M has repeated eigenvalues and may not be diagonalizable. With this choice of γ, we have

$$M = \begin{bmatrix} 2 & 2 \\ -\frac{1}{2} & 0 \end{bmatrix} \quad \text{and} \quad \lambda_{1,2} = 1,$$

so that M is obviously non-diagonalizable. Nonetheless, we can still find a basis for the semi-stable invariant subspace of M by solving the equations

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

which yields,

$$x = 1 \quad \text{and} \quad y = -\frac{1}{2}.$$

The semi-stabilizing solution to the DARE is therefore given by

$$p = xy^{-1} = -2,$$

which corresponds to the solution that can be obtained from solving (7.7.5) when $\gamma^2 = 4$. The spectral factorization of $\Sigma(z)$ now becomes

$$\Sigma(z) = \frac{z - 1}{z - 1/2} \cdot 2 \cdot \frac{z^{-1} - 1}{z^{-1} - 1/2}.$$

■

Our next example concerns the case where $\Sigma(z)$, and hence R_e, is singular. This example shows that R_e may be singular even when $\{F, G, H\}$ is minimal, and hence serves to demonstrate that, unlike the positive case, it is very difficult to give necessary and sufficient conditions for the invertibility of R_e in the indefinite case.

Example 4 (Singular $\Sigma(z)$) Consider the SDARE

$$\left\{\begin{array}{rcl} P &=& FPF^* + GQG^* - K_p R_e K_p^* \\ FPH^* &=& K_p R_e \\ R_e &=& R + HPH^* \end{array}\right. \tag{7.7.7}$$

where

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \;,\; G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;,\; H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \;,\; S = 0 \;,\; R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to check that one solution to the above SDARE is given by

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;,\; R_e = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \;,\; K_p = \begin{bmatrix} 1/2 & 0 \\ 0 & \cos\theta \end{bmatrix}$$

where θ is arbitrary. Note that this solution is stabilizing since

$$F - K_p H = \begin{bmatrix} 1/2 & 0 \\ 0 & -\cos\theta \end{bmatrix},$$

is stable for all θ.

This example is of interest since it shows that in the indefinite case, even when $\{F, G, H\}$ is minimal, the solution to the DARE may result in a singular R_e. The reason is that the Popov function associated with this SDARE is singular. Indeed,

$$\Sigma(z) \;=\; \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1} \\ I \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} z-1 & 0 \\ 0 & z \end{bmatrix}^{-1} & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} z^{-1}-1 & 0 \\ 0 & z^{-1} \end{bmatrix}^{-1} \\ I \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \cdot \frac{1}{(z-1)(z^{-1}-1)} + 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

■

Example 5 (DARE Without Semi-Stabilizing Solution) Consider the DARE with $F = 1$, $G = 1$, $H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Q = 3$, $R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 0 \end{bmatrix}$. Using the fact that the general DARE (with $S = 0$) can be written as

$$P = F(I + PH^*R^{-1}H)^{-1}PF^* + GQG^*,$$

allows us to write the DARE, in this special case, as

$$P = 4P + 3,$$

from which we infer $P = -1$. Thus the DARE has a unique (Hermitian solution). It is now straightforward to see that,

$$R_e = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix} \quad , \quad R_e^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \quad , \quad K_p = \begin{bmatrix} 2 & -2 \end{bmatrix},$$

from which we can infer,

$$F_p = F - K_pH = F - 0 = 2.$$

Therefore the solution to the DARE is not (semi)stabilizing. Indeed, although $\{F, H\}$ is detectable, and although the DARE has a Hermitian solution, this solution is not stable. This, of course, means that the Popov function,

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{3}{(z-2)(z^{-1}-2)} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \frac{3}{(z-2)(z^{-1}-2)} & \frac{3}{(z-2)(z^{-1}-2)} \\ \frac{3}{(z-2)(z^{-1}-2)} & -1 + \frac{3}{(z-2)(z^{-1}-2)} \end{bmatrix} \end{aligned}$$

does not admit a proper canonical Hermitian factorization. Using the above solution to the DARE, the proper factorization that it does admit is given by

$$\Sigma(z)=\begin{bmatrix}\frac{z-4}{(z-2)(z^{-1}-2)} & \frac{2}{(z-2)(z^{-1}-2)}\\ \frac{-2}{(z-2)(z^{-1}-2)} & \frac{z}{(z-2)(z^{-1}-2)}\end{bmatrix}\begin{bmatrix}0 & -1\\ -1 & -2\end{bmatrix}\begin{bmatrix}\frac{z^{-1}-4}{(z-2)(z^{-1}-2)} & \frac{-2}{(z-2)(z^{-1}-2)}\\ \frac{2}{(z-2)(z^{-1}-2)} & \frac{z^{-1}}{(z-2)(z^{-1}-2)}\end{bmatrix},$$

which is a "dual" canonical factorization. [Note also that, $\begin{bmatrix}1\\0\end{bmatrix}$, the stable invariant subspace of

$$M=\begin{bmatrix}\frac{1}{2} & -\frac{3}{2}\\ 0 & 2\end{bmatrix},$$

does not have an invertible $\mathcal{T}_{21}$, whereas the antistable invariant subspace, $\begin{bmatrix}1\\-1\end{bmatrix}$ does.]

■

Example 6 (DARE Without Solution) Consider the DARE with $F=1$, $G=1$, $H=\begin{bmatrix}1\\1\end{bmatrix}$, $Q=1$, $R=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and $S=\begin{bmatrix}0 & 0\end{bmatrix}$. Using the fact that the general DARE (with $S=0$) can be written as

$$P=F(I+PH^*R^{-1}H)^{-1}PF^*+GQG^*,$$

allows us to write the DARE, in this special case, as

$$P=P+1,$$

from which we infer that the DARE has no solution! This implies that, despite the fact that $\{F,H\}$ is detectable, the Popov function,

$$\begin{aligned}\Sigma(z) &= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{(z-1)(z^{-1}-1)}\begin{bmatrix}1 & 1\end{bmatrix}\\ &= \begin{bmatrix}1+\frac{1}{(z-1)(z^{-1}-1)} & \frac{1}{(z-1)(z^{-1}-1)}\\ \frac{1}{(z-1)(z^{-1}-1)} & -1+\frac{1}{(z-1)(z^{-1}-1)}\end{bmatrix}\end{aligned}$$

does not admit any proper factorization. [Note that in this case, the only invariant subspace of the Hamiltonian matrix,

$$M = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which does not have an invertible $\mathcal{T}_{21}$.]

■

7.8 Conclusion

In this chapter we studied the discrete-time algebraic Riccati equation (DARE), and showed that the existence of solutions to the DARE, or more precisely the system of discrete-time algebraic Riccati equations (SDARE), is equivalent to the existence of a proper factorization of the associated Popov function. Additional properties of the DARE solution, such as being Hermitian or stabilizing, were then related to additional properties of the factorization of the Popov function. A method for computing the solution to the DARE, and, in fact, for checking its existence was also given in terms of invariant subspaces of the so-called Hamiltonian matrix. We also particularized these results to some important special cases that are encountered in LQR and H^∞ estimation and control. Finally, we should mention that the results presented here are extensions of some wellknown results on discrete-time algebraic Riccati equations with positive (semi)definite coefficient matrices to the indefinite (or Krein space) setting.

7.A Structure of Z_1 in Theorem 7.4.1

In this Appendix we shall show that if Z_1 satisfies the Lyapunov equation

$$Z_1 = F Z_1 F^* + T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^*, \tag{7.A.1}$$

where

$$F = T_a \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} T_a^{-1},$$

and

$$F = T_b \begin{bmatrix} F_b & 0 & F_{b,13} & 0 \\ F_{b,21} & F_{b,o} & F_{b,23} & F_{b,24} \\ 0 & 0 & F_{b,c} & 0 \\ 0 & 0 & F_{b,43} & F_{b,co} \end{bmatrix} T_b^{-1},$$

then Z_1 has the form

$$Z_1 = T_a \begin{bmatrix} Z_{ab} & Z_{ab,o} & 0 & 0 \\ Z_{ba,o} & Z_{ab,oo} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} T_b^*, \tag{7.A.2}$$

where

$$\begin{bmatrix} Z_{ab} & Z_{ab,o} \\ Z_{ba,o} & Z_{ab,oo} \end{bmatrix} = \begin{bmatrix} F_a & 0 \\ F_{a,21} & F_{a,o} \end{bmatrix} \begin{bmatrix} Z_{ab} & Z_{ab,o} \\ Z_{ba,o} & Z_{ab,oo} \end{bmatrix} \begin{bmatrix} F_b^* & F_{b,21}^* \\ 0 & F_{b,o}^* \end{bmatrix} + \begin{bmatrix} K_a \\ X_a \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \end{bmatrix}^*. \tag{7.A.3}$$

To this end, let us pre- and post-multiply (7.A.1) by T_a^{-1} and T_b^{-*}, respectively, to obtain

$$T_a^{-1}Z_1T_b^{-*} = \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} Z_1 \begin{bmatrix} F_b & 0 & F_{b,13} & 0 \\ F_{b,21} & F_{b,o} & F_{b,23} & F_{b,24} \\ 0 & 0 & F_{b,c} & 0 \\ 0 & 0 & F_{b,43} & F_{b,co} \end{bmatrix} + \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* . \tag{7.A.4}$$

If we now partition $T_a^{-1}Z_1T_b^{-*}$ as

$$T_a^{-1}Z_1T_b^{-*} = \begin{bmatrix} Z_1^{11} & Z_1^{12} \\ Z_1^{21} & Z_1^{22} \end{bmatrix},$$

and define

$$F_a^{11} = \begin{bmatrix} F_a & 0 \\ F_{a,21} & F_{a,o} \end{bmatrix}, \; F_a^{12} = \begin{bmatrix} F_{a,13} & 0 \\ F_{a,23} & F_{a,24} \end{bmatrix}, \; F_a^{22} = \begin{bmatrix} F_{a,c} & 0 \\ F_{a,43} & F_{a,co} \end{bmatrix}$$

then we may write (7.A.4) as

$$\begin{bmatrix} Z_1^{11} & Z_1^{12} \\ Z_1^{21} & Z_1^{22} \end{bmatrix} = \begin{bmatrix} F_a^{11} & F_a^{12} \\ 0 & F_a^{22} \end{bmatrix} \begin{bmatrix} Z_1^{11} & Z_1^{12} \\ Z_1^{21} & Z_1^{22} \end{bmatrix} \begin{bmatrix} F_b^{11} & F_b^{12} \\ 0 & F_b^{22} \end{bmatrix}^* + \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}, \tag{7.A.5}$$

where we have made similar definitions for F_b^{11}, F_b^{12} and F_b^{22}, and where U is given by

$$U = \begin{bmatrix} K_a \\ X_a \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \end{bmatrix}^* .$$

Now (7.A.5) may be rewritten as

$$\begin{bmatrix} Z_1^{11} & Z_1^{12} \\ Z_1^{21} & Z_1^{22} \end{bmatrix} = \begin{bmatrix} F_a^{11}Z_1^{11}F_b^{11*} + F_a^{12}Z_1^{21}F_b^{11*} + F_a^{11}Z_1^{11}F_b^{12*} + F_a^{12}Z_1^{22}F_b^{12*} + U & \times \\ F_a^{22}Z_1^{21}F_b^{11*} + F_a^{22}Z_1^{22}F_b^{12*} & F_a^{22}Z_1^{22}F_b^{22*} \end{bmatrix} . \tag{7.A.6}$$

Equating the $(2,2)$ block entries in the above equality yields

$$Z_1^{22} = F_a^{22}Z_1^{22}F_b^{22*},$$

from which we conclude (since both F_a^{22} and F_b^{22} are stable) that

$$Z_1^{22} = 0. \tag{7.A.7}$$

Using the above result and equating the $(2,1)$ block entries yields

$$Z_1^{21} = F_a^{22} Z_1^{21} F_b^{11*},$$

from which we conclude (since F_a^{22} is stable and F_b^{11} is marginally stable) that

$$Z_1^{21} = 0. \tag{7.A.8}$$

A similar argument shows that

$$Z_1^{12} = 0. \tag{7.A.9}$$

Eqs. (7.A.7-7.A.9) show that Z_1 has the desired structure given in (7.A.2). Finally, (7.A.3) follows from equating the $(1,1)$ block entries in (7.A.6).

■