# Chapter 8

# Asymptotic Behaviour

In this chapter we shall focus on the behaviour of the Riccati recursion as time progresses to infinity. In order to do so, we shall confine our attention to time-invariant state-space models. Our main interest is to find conditions under which, for a given initial condition, the solution to the Riccati recursion converges to a solution of the DARE. The main result uses an identity that relates the solution of the Riccati recursion for one initial condition to the solution for another initial condition, and states that if a certain inertia condition is met, then the Riccati recursion converges to the unique stabilizing solution (assuming such a solution exists) of the DARE. In the general case, the aforementioned inertia conditions need to be recursively checked, however, in some special cases they may be reduced to more simple and more explicit requirements on the initial condition. In particular, when the coefficient matrices of the Riccati recursion are positive semi-definite, we can guarantee the convergence of the Riccati recursion for some indefinite, and even negative semi-definite, initial conditions (provided they are bounded below by a certain negative semi-definite matrix). Moreover, in the case frequently encountered in $H^\infty$ filtering and control, we can guarantee convergence for all positive semi-definite initial conditions that are less than or equal to the unique positive semi-definite solution of a related Lyapunov equation. Finally, we mention some implications of the results obtained here to different problems in which these Riccati equations occur.

## 8.1 The Riccati Recursion

In this section we shall consider the discrete-time algebraic Riccati recursion (DARR)

$$P_{i+1} = FP_iF^* + GQG^* - (FP_iH^* + GS)(R + HP_iH^*)^{-1}(FP_iH^* + GS), \quad P_0 = \Pi_0 \tag{8.1.1}$$

where $F \in \mathcal{C}^{n\times n}$, $G \in \mathcal{C}^{n\times m}$ and $H \in \mathcal{C}^{p\times n}$ are given, $Q = Q^* \in \mathcal{C}^{m\times m}$, $R = R^* \in \mathcal{C}^{p\times p}$ and $S \in \mathcal{C}^{m\times p}$ are known, and it is assumed that the Hermitian matrices

$$R \quad \text{and} \quad \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix},$$

are nonsingular. The (possibly Hermitian) sequence of $n \times n$ matrices $\{P_i\}$ that satisfies (8.1.1) is referred to as the solution to the DARR with given initial condition, $P_0 = \Pi_0$.

As before, we shall often find it convenient to introduce the notations

$$K_{p,i} = (FP_iH^* + GS)(R + HP_iH^*)^{-1} \quad \text{and} \quad R_{e,i} = R + HP_iH^*, \tag{8.1.2}$$

and, when $P_i$ is Hermitian, to rewrite the DARR (8.1.1) as

$$P_{i+1} = FP_iF^* - K_{p,i}R_{e,i}K_{p,i}^* + GQG^*. \tag{8.1.3}$$

When, $P_i$, the solution to the DARR is not Hermitian, we can define

$$K_{q,i} = (FP_i^*H^* + GS)(R + HP_i^*H^*)^{-1}, \tag{8.1.4}$$

and rewrite the DARR as

$$P_{i+1} = FP_iF^* - K_{p,i}R_{e,i}K_{q,i}^* + GQG^*. \tag{8.1.5}$$

The main question of interest in this section is to clarify conditions under which, for a given initial condition $P_0$, the Riccati variable $P_i$ converges to $P$, some solution of the DARE (7.2.1). This question, of the asymptotic behaviour of the Riccati recursion, is important since it gives conditions under which time-varying least-squares (or $H^\infty$) controllers and filters converge to time-invariant ones.

In this section we shall give what we believe is a very direct approach to establishing the convergence of the Riccati recursion. Moreover, to the best of our knowledge, the results obtained here are more general than those to have appeared in the literature and subsume as special cases all of the earlier given results [Wil71b, LK76, Kuc72, Won68a, BRC72, CW81]. Since the section has quite a few results, it will be useful to begin with a brief overview of what shall be presented.

## 8.2 Overview of Results

We begin by proving two important identities relating the solution of the Riccati recursion for one initial condition to the solution of the Riccati recursion for another initial condition. The first of these is a local identity (see Lemma 8.3.1) while the second is a global identity (see Lemma 8.3.2). The result of Lemma 8.3.2 is then used in Sec. 8.4 to compare the solution of the Riccati recursion for an arbitrary initial condition, $P_0$, with any solution of the corresponding DARE. The ensuing result is given in Theorem 8.4.1 where it is shown that if a certain sequence of matrices (that depend on $P_0$) are uniformly bounded for all $i \geq 0$, then the solution to the Riccati recursion will *exponentially* converge to the unique stabilizing solution of the DARE (assuming such a solution exists). A sufficient condition for convergence turns out to be the condition that the matrices,

$$R_{e,i} = R + HP_iH^*,$$

have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$, where $P_i$ is the solution of the Riccati recursion with initial condition $P_0$.

Theorems 8.5.1 and 8.5.2 prove the convergence of the zero-initial-condition Riccati recursion (corresponding to $P_0 = 0$) for the positive case and the case where $Q - SR^{-1}S^* > 0$, respectively. The proof in both cases follows from showing that the sequence of matrices,

$$R_{e,i}^0 = R + HP_i^0H^*,$$

where $P_i^0$ is the zero-initial-condition Riccati solution, has constant inertia for all $i \geq 0$.

The fact that we can establish the convergence of $\{P_i^0\}$ to $P$, the unique stabilizing solution of the DARE, allows us to obtain more simple and more explicit conditions for convergence than the aforementioned inertia condition on the $\{R_{e,i}\}$. Indeed we can use Lemma 8.3.2 to compare $\{P_i\}$, the solution of the Riccati recursion for an arbitrary initial condition, to $\{P_i^0\}$, the zero initial condition solution, for which we have already established convergence. The relationship between $\{P_i\}$ and $\{P_i^0\}$ can be expressed more concisely in terms of quantities defined via the *dual* Riccati recursion

$$P_{i+1}^a = F^* P_i^a F + H^* R^{-1} H - F^* P_i^a G (Q^{-1} + G^* P_i^a G)^{-1} G^* P_i^a F, \qquad (8.2.1)$$

defined in Sec. 8.6.1. The exact relationship is given by

$$P_{i+1} - P_{i+1}^0 = \Phi_p^0(i+1,0) \left[ I + P_0 P_{i+1}^{a,0} \right]^{-1} \Pi_0 \Phi_p^0(i+1,0)^*, \qquad (8.2.2)$$

where $P_i^{a,0}$ is the zero initial condition solution to the dual Riccati recursion (8.2.1) and $\Phi_p^0(i+1,0)$ is a certain state transition matrix. Since it can be shown that, under certain stabilizability conditions, the state transition matrix $\Phi_p^0(i+1,0)$ tends to zero as $i \to \infty$, to show the convergence of $P_i$ to $P_i^0$ (and hence to $P$) we need only show that the matrix $\left[ I + P_0 P_{i+1}^{a,0} \right]^{-1}$ is uniformly bounded for all $i$. In the positive case (where $R > 0$ and $Q - SR^{-1}S^* > 0$) we establish that a sufficient condition for this is that the initial condition $P_0$ be such that

$$I + (P^a)^{*/2} P_0 (P^a)^{1/2} > 0, \qquad (8.2.3)$$

where $(P^a)^{1/2}(P^a)^{*/2} = P^a$, is the unique positive semi-definite solution to the dual DARE. The above result, whose precise statement is given in Theorem 8.7.1, gives, in the positive case, a very general description of the basin of attraction of the stabilizing solution $P$ to the Riccati recursion. Note that convergence for $P_0 \geq 0$ follows immediately from (8.2.3). However, the condition (8.2.3) also allows for some indefinite (and even negative semi-definite) initial conditions $P_0$. Indeed, under a certain controllability assumption, it can be shown that $P^a$ is invertible so that (8.2.3) can be replaced by the more revealing condition

$$P_0 > -(P^a)^{-1}. \qquad (8.2.4)$$

Finally, in the case where $Q - SR^{-1}S^* > 0$ it does not seem to be possible to replace the condition that $R_{e,i}$ should have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$, with a single condition such as (8.2.3). [Nonetheless, the inertia condition on $R_{e,i}$ is quite intriguing since it is equivalent to the condition required for the existence of $H^\infty$ filters and full-information controllers.] However, it is still possible to give a simple characterization of the basin of attraction of the stabilizing solution, $P$, of the DARE in this case as well. Indeed, Theorem 8.7.2 shows that if the initial condition is chosen such that

$$0 \leq P_0 \leq \Pi,$$

where $\Pi$ is the unique positive semidefinite solution to the Lyapunov equation, $\Pi = F\Pi F^* + GQG^*$, then $\{P_i\}$ converges to $P$.

## 8.3 Solutions to the Riccati Recursion for Different Initial Conditions

We first give a certain algebraic identity concerning the Riccati recursion (8.1.1). These identities are wellknown [Nis67, MPG88, dS89], and are very useful for obtaining comparison results for solutions to the Riccati recursion.

**Lemma 8.3.1 (Riccati Solutions for Different Initial Conditions)** *Suppose $P_i^{(1)}$ and $P_i^{(2)}$ are two solutions to the discrete-time Riccati recursion with the same $\{F, G, H\}$ and $\{Q, R, S\}$ matrices, but with different initial conditions $\Pi_0^{(1)}$ and $\Pi_0^{(2)}$, respectively. Then we have the following identities*

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = F_{p,i}^{(1)}(P_i^{(2)} - P_i^{(1)})F_{p,i}^{(2)*} \tag{8.3.1}$$

*and*

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = F_{p,i}^{(1)}\left[(P_i^{(2)} - P_i^{(1)}) - (P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(2)})^{-1}H(P_i^{(2)} - P_i^{(1)})\right]F_{p,i}^{(1)*}, \tag{8.3.2}$$

*where we have defined*

$$F_{p,i}^{(m)} = F - K_{p,i}^{(m)} H, \quad K_{p,i}^{(m)} = (FP_i^{(m)} H^* + GS)(R_{e,i}^{(m)})^{-1} \quad \textit{and} \quad R_{e,i}^{(m)} = R + HP_i^{(m)} H^*, \quad m = 1, 2.$$

**Remark:** Note that (8.3.1) gives a simpler expression for $P_{i+1}^{(2)} - P_{i+1}^{(1)}$ than (8.3.2). However, (8.3.2) is often more useful since it gives a *symmetric* representation for $P_{i+1}^{(2)} - P_{i+1}^{(1)}$. In the following sections we shall make use of both identities.

**Proof of Lemma 8.3.1:** The proof involves only algebraic manipulations. We shall outline the steps. We first mention that in what follows we shall, without loss of generality, assume that $S = 0$. It is now straightforward to verify the identities

$$R_{e,i}^{(2)} - R_{e,i}^{(1)} = H(P_i^{(2)} - P_i^{(1)})H^*, \tag{8.3.3}$$

and

$$K_{p,i}^{(2)} - K_{p,i}^{(1)} = F_{p,i}^{(1)}(P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(2)})^{-1}. \tag{8.3.4}$$

Now subtracting the two Riccati recursions for $P_i^{(1)}$ and $P_i^{(2)}$ yields

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = F(P_i^{(2)} - P_i^{(1)})F^* - K_{p,i}^{(2)} R_{e,i}^{(2)} K_{p,i}^{(2)*} + K_{p,i}^{(1)} R_{e,i}^{(1)} K_{p,i}^{(1)*}. \tag{8.3.5}$$

Now applying the identities (8.3.3)-(8.3.4), along with a rearrangement of the formula $F_{p,i} = F - K_{p,i}H$, to the above expression, we may write

$$\begin{aligned} P_{i+1}^{(2)} - P_{i+1}^{(1)} &= (F_{p,i}^{(1)} + K_{p,i}^{(1)} H)(P_i^{(2)} - P_i^{(1)})(F_{p,i}^{(2)} + K_{p,i}^{(2)} H)^* \\ &\quad -(F_{p,i}^{(1)}(P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(2)})^{-1})R_{e,i}^{(2)} K_{p,i}^{(2)*} + K_{p,i}^{(1)} R_{e,i}^{(1)} (F_{p,i}^{(2)}(P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(1)})^{-1})R_{e,i}^{(1)}, \end{aligned}$$

which, thanks to nice cancellations, yields (8.3.1).

To prove (8.3.1), we rewrite (8.3.5) as

$$\begin{aligned} P_{i+1}^{(2)} - P_{i+1}^{(1)} &= (F_{p,i}^{(1)} + K_{p,i}^{(1)} H)(P_i^{(2)} - P_i^{(1)})(F_{p,i}^{(1)} + K_{p,i}^{(1)} H)^* \\ &\quad -(F_{p,i}^{(1)}(P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(2)})^{-1})(R_{e,i}^{(2)}(F_{p,i}^{(1)}(P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(2)})^{-1})^* \\ &\quad +K_{p,i}^{(1)}(R_{e,i}^{(2)} - H(P_i^{(2)} - P_i^{(1)})H^*)K_{p,i}^{(1)*}, \end{aligned}$$

and simplify.

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The next identity that we shall establish is just the statement of the fact that if we know a solution to the Riccati recursion for one initial condition, then under certain conditions, solutions for other initial conditions can be expressed in terms of the first solution. Although formulas of this type exist in continuous-time (see [San59, Rei72]), we have not encountered their discrete-time analogs in the literature. We have already provided a *local* identity relating solutions of the Riccati recursion for different initial conditions in Lemma 8.3.1. What we shall now present is a *global* identity. The precise statement follows.

**Lemma 8.3.2 (Riccati Solutions for Different Initial Conditions)** *Suppose $P_i^{(1)}$ and $P_i^{(2)}$ are two solutions to the discrete-time Riccati recursion (8.1.1) with the same $\{F, G, H\}$ and $\{Q, R, S\}$ matrices, but with different initial conditions $\Pi_0^{(1)}$ and $\Pi_0^{(2)}$, respectively. We then have the following identity:*

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \Phi_p^{(1)}(i+1, 0)\left[I + (\Pi_0^{(2)} - \Pi_0^{(1)})\mathcal{O}_i^{(1)}\right]^{-1}(\Pi_0^{(2)} - \Pi_0^{(1)})\Phi_p^{(1)}(i+1, 0)^*, \quad (8.3.6)$$

*where*

$$\Phi_p^{(1)}(i, 0) = \begin{cases} \prod_{j=0}^{i-1} F_{p,j}^{(1)} & i > 0 \\ I & i = 0 \end{cases} \quad (8.3.7)$$

*is the state transition matrix of $F_{p,j}^{(1)} = F - K_{p,j}^{(1)} H$, and*

$$\mathcal{O}_i^{(1)} = \sum_{j=0}^{i} \Phi_p^{(1)}(j, 0)^* H^* (R_{e,j}^{(1)})^{-1} H \Phi_p^{(1)}(j, 0), \quad (8.3.8)$$

*is the observability Gramian of $\{F_{p,j}^{(1)}, (R_{e,j}^{(1)})^{-1/2} H\}$.*

**Proof:** We shall prove (8.3.6) by induction. Define $\delta P_i \triangleq P_i^{(2)} - P_i^{(1)}$. Then for $i = 0$, using (8.3.2), we have

$$\begin{aligned}
P_1^{(2)} - P_1^{(1)} &= F_{p,0}^{(1)}\left[\delta P_0 - \delta P_0 H^* (R_{e,0}^{(2)})^{-1} H \delta P_0\right] F_{p,0}^{(1)*} \\
&= \Phi_p^{(1)}(1, 0)\left[I - \delta P_0 H^* (R_{e,0}^{(2)})^{-1} H\right] \delta P_0 \Phi_p^{(1)}(1, 0)^* \\
&= \Phi_p^{(1)}(1, 0)\left[I - \delta P_0 H^* (R_{e,0}^{(1)} + H \delta P_0 H^*)^{-1} H\right] \delta P_0 \Phi_p^{(1)}(1, 0)^* \\
&= \Phi_p^{(1)}(1, 0)\left[I + \delta P_0 H^* (R_{e,0}^{(1)})^{-1} H\right]^{-1} \delta P_0 \Phi_p^{(1)}(1, 0)^* \\
&= \Phi_p^{(1)}(1, 0)\left[I + \delta P_0 \mathcal{O}_0^{(1)}\right]^{-1} \delta P_0 \Phi_p^{(1)}(1, 0)^*
\end{aligned}$$

as desired.

Now suppose that (8.3.6) is true for $i$. We shall show that the identity is true for $i+1$ as well. Indeed, using (8.3.2) and the above arguments we have

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = F_{p,i}^{(1)}\left[\delta P_i - \delta P_i H^*(R_{e,i}^{(2)})^{-1} H \delta P_i\right] F_{p,i}^{(1)*} = F_{p,i}^{(1)}\left[I + \delta P_i H^*(R_{e,i}^{(1)})^{-1} H\right]^{-1} \delta P_i F_{p,i}^{(1)*}. \tag{8.3.9}$$

On the other hand,

$$\begin{aligned}
I + \delta P_i H^*(R_{e,i}^{(1)})^{-1} H &= I + \Phi_p^{(1)}(i,0)\left[I + \delta P_0 \mathcal{O}_{i-1}^{(1)}\right]^{-1} \delta P_0 \Phi_p^{(1)}(i,0)^* H^* (R_{e,i}^{(1)})^{-1} H \\
&= I + \Phi_p^{(1)}(i,0)\left[I + \delta P_0 \mathcal{O}_{i-1}^{(1)}\right]^{-1} \delta P_0 \underbrace{\Phi_p^{(1)}(i,0)^* H^* (R_{e,i}^{(1)})^{-1} H \Phi_p^{(1)}(i,0)}_{\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)}} \Phi_p^{(1)}(i,0)^{-1} \\
&= I + \Phi_p^{(1)}(i,0)\left[I + \delta P_0 \mathcal{O}_{i-1}^{(1)}\right]^{-1} \delta P_0 (\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)}) \Phi_p^{(1)}(i,0)^{-1} \\
&= \Phi_p^{(1)}(i,0)\left[I + (I + \delta P_0 \mathcal{O}_{i-1}^{(1)})^{-1} \delta P_0 (\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)})\right] \Phi_p^{(1)}(i,0)^{-1}.
\end{aligned}$$

Using the last of the above expressions in (8.3.9) yields

$$\begin{aligned}
P_{i+1}^{(2)} - P_{i+1}^{(1)} &= \Phi_p^{(1)}(i+1,0)\left[I + (I + \delta P_0 \mathcal{O}_{i-1}^{(1)})^{-1} \delta P_0 (\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)})\right]^{-1} \Phi_p^{(1)}(i,0)^{-1} \delta P_i F_{p,i}^{(1)*} \\
&= \Phi_p^{(1)}(i+1,0)\left[I + (I + \delta P_0 \mathcal{O}_{i-1}^{(1)})^{-1} \delta P_0 (\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)})\right]^{-1} (I + \delta P_0 \mathcal{O}_{i-1}^{(1)})^{-1} \delta P_0 \Phi_p^{(1)}(i+1,0)^* \\
&= \Phi_p^{(1)}(i+1,0)\left[I + \delta P_0 \mathcal{O}_{i-1}^{(1)} + \delta P_0 (\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)})\right]^{-1} \delta P_0 \Phi_p^{(1)}(i+1,0)^* \\
&= \Phi_p^{(1)}(i+1,0)\left[I + \delta P_0 \mathcal{O}_i^{(1)}\right]^{-1} \delta P_0 \Phi_p^{(1)}(i+1,0)^*,
\end{aligned}$$

which is the desired result.

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The result of Lemma 8.3.2 will prove to be very useful in establishing convergence results for the Riccati recursion (8.1.1). To see why, suppose is $P$ is some solution to the DARE (7.2.1). Then using (8.3.6), we may write

$$P_{i+1} - P = F_p^{i+1}\left[I + (P_0 - P)\mathcal{O}_i^p\right]^{-1}(P_0 - P)F_p^{(i+1)*}, \tag{8.3.10}$$

where $P_i$ is the solution to the Riccati recursion with initial condition $P_0$, and $\mathcal{O}_i^p$ is the observability Gramian of the pair $\{F_p, R_e^{-1/2}H\}$, and satisfies the recursion

$$\mathcal{O}_{i+1}^p = F_p^* \mathcal{O}_i^p F_p + H^* R_e H, \qquad \mathcal{O}_{-1}^p = 0.$$

Now if $P$ is chosen as the stabilizing solution to the DARE (assuming such a solution exists), then the matrix $F_p = F - K_p H$ is stable and we have $\lim_{i\to\infty} F_p^i = 0$.

Then using (8.3.10), we see that if the initial condition $P_0$ is such that the matrix $[I+(P_0-P)\mathcal{O}_i^p]^{-1}(P_0-P)$ remains uniformly bounded for all $i$, then $P_i$ will converge to $P$, *i.e.*,

$$\lim_{i\to\infty} P_{i+1}-P=0.$$

The above line of reasoning will be used in the next section to establish general convergence results for the Riccati recursion.

## 8.4 Some General Convergence Results

In this section we shall obtain some general results concerning the convergence of the Riccati recursion (8.1.1). We shall essentially give requirements on the initial condition, $P_0$, such that the solution of the Riccati recursion converges to the unique stabilizing solution (when a stabilizing solution exists) of the discrete-time algebraic Riccati equation. Although the requirements on $P_0$ may be difficult to verify in the general case, in later sections we shall see that, when specialized to some very important cases of interest in least-squares and $H^\infty$ control and estimation, they result in easy to verify requirements and yield basins of attraction for the initial condition, $P_0$, that are more general than those currently available in the literature.

The results presented in this section all use the identity (8.3.10). However, it will be useful to introduce a more symmetric representation of (8.3.10). To this end, let us introduce the generalized square-root factorization of the (possibly indefinite) Hermitian matrix $P_0-P$, as follows

$$P_0-P=L_0JL_0^*, \tag{8.4.1}$$

where $J$ is a signature matrix of appropriate dimensions that represents the inertia of $P_0-P$. Then using the identity

$$[I+L_0JL_0^*\mathcal{O}_i^p]^{-1}L_0JL_0^*=L_0[J+L_0^*\mathcal{O}_i^pL_0]^{-1}L_0^*,$$

we may write (8.3.10) as

$$P_{i+1}-P=F_p^{i+1}L_0[J+L_0^*\mathcal{O}_i^pL_0]^{-1}L_0^*F_p^{(i+1)*}, \tag{8.4.2}$$

which is the form that we shall more often use.

**Theorem 8.4.1 (A General Convergence Result)** *Consider the discrete-time Riccati recursion*

$$P_{i+1} = FP_iF^* + GQG^* - (FP_iH^* + GS)(R + HP_iH^*)^{-1}(FP_iH^* + GS), \quad (8.4.3)$$

*with initial condition, $P_0$, and suppose that the associated DARE (7.2.1) has a stabilizing solution, $P$. Then if the initial condition $P_0$ is chosen such that the matrices*

$$[J + L_0^*\mathcal{O}_i^p L_0]^{-1},$$

*are uniformly bounded for all $i \geq 0$, where*

$$P_0 - P = L_0 J L_0^*,$$

*and $\mathcal{O}_i^p$ satisfies the recursion*

$$\mathcal{O}_{i+1}^p = F_p^*\mathcal{O}_i^p F_p + H^*R_e^{-1}H, \qquad \mathcal{O}_{-1}^p = 0,$$

*then $P_i$ converges to, $P$, the unique stabilizing solution of the DARE. Moreover, we have*

$$\|P_i - P\| \leq \lambda^{2i} m, \quad (8.4.4)$$

*where $\|\cdot\|$ denotes the spectral radius,*

$$\lambda = \|F - K_pH\| \leq 1, \quad (8.4.5)$$

*and*

$$m = \sup_{i\geq 0} \left\| L_0 \left[J + L_0^*\mathcal{O}_i^p L_0\right]^{-1} L_0^* \right\|, \quad (8.4.6)$$

i.e., *the convergence of $P_i$ to $P$ is exponential.*

*Two equivalent conditions that ensure the boundedness of $[J + L_0^*\mathcal{O}_i^p L_0]^{-1}$ are the following:*

*(a) The matrices $J + L_0^*\mathcal{O}_i^p L_0$ have constant inertia (equal to the inertia of $J$) for all $i \geq 0$.*

*(b) The matrices $R_{e,i} = R + HP_iH^*$ have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$.*

**Remark:** The condition given in the above Theorem for convergence to the stabilizing solution of the DARE is essentially that the sequence of matrices $T_i^{-1} = [J + L_0^* \mathcal{O}_i^p L_0]^{-1}$ be uniformly bounded for all $i \geq 0$. This condition is interesting since it only depends on the initial condition through the matrices $L_0$ and $J$ (the $\mathcal{O}_i^p$ do not depend on $P_0$). However, this condition may be quite difficult to verify in the general case. The sufficient conditions (a) and (b) for boundedness seem to be much more restrictive, since instead of requiring that the $T_i$ be uniformly nonsingular for all $i$, they require that the $T_i$ have constant inertia for all $i$. Despite this fact, as we shall see in subsequent sections, the conditions (a) and (b) are much easier to verify, and are still quite general since they yield conditions for convergence that are quite more general than those that have appeared in the literature so far.

**Proof of Theorem 8.4.1:** To prove the first part we need only consider (8.4.2). Since $P$ is stabilizing, so that $F_p = F - K_p H$ is stable and $\lim_{i\to\infty} F_p^i = 0$, and since $[J + L_0^* \mathcal{O}_i^p L_0]^{-1}$ is uniformly bounded for all $i \geq 0$, we immediately deduce that

$$\lim_{i\to\infty} P_i - P = 0,$$

as desired.

To show the exponential convergence of $P_i$, we can use (8.4.2) to compute $\|P_i - P\|$. Thus, using the submultiplicative property of the spectral radius ($\|AB\| \leq \|A\| \cdot \|B\|$), we have

$$\|P_i - P\| = \|F - K_p H\|^i \cdot \left\| L_0 \left[J + L_0^* \mathcal{O}_i^p L_0\right]^{-1} L_0^* \right\| \cdot \|F - K_p H\|^i.$$

The desired result (8.4.4) now follows from (8.4.5) and (8.4.6).

Now the fact that condition (a) leads to uniformly bounded $[J + L_0^* \mathcal{O}_i^p L_0]^{-1}$ for all $i$ is obvious, since if the $J + L_0^* \mathcal{O}_i^p L_0$ have constant inertia (equal to the inertia of $J$) for all $i$, then they are uniformly nonsingular as well. Therefore, what remains to be shown is that the conditions (a) and (b) are equivalent. This requires somewhat more effort.

To this end, let us first write

$$P_i - P = L_i D_i L_i^*,$$

where $D_i$ is a nonsingular Hermitian matrix of the appropriate dimensions. Now using (8.3.2) and the above equation, we may write

$$P_{i+1} - P = F_p L_i \left[D_i - D_i L_i^* H R_{e,i}^{-1} H^* L_i^* D_i\right]^{-1} L_i^* F_p^*, \tag{8.4.7}$$

from which we may infer

$$L_{i+1} = F_p L_i = F_p^{i+1} L_0,$$

and

$$\begin{aligned} D_{i+1} &= D_i - D_i L_i^* H R_{e,i}^{-1} H^* L_i^* D_i \\ &= D_i - D_i L_i^* H (R_{e,i} + H L_i D_i L_i^* H^*)^{-1} H^* L_i^* D_i \\ &= \left[D_i^{-1} + H^* R_{e,i}^{-1} H\right]^{-1}. \end{aligned}$$

Thus, (8.4.7) may be written as

$$P_{i+1} - P = F_p^{i+1} L_0 D_{i+1} L_0^* F_p^{(i+1)*}.$$

Comparing the above expression with (8.4.2), we note that the matrices $[J + L_0^* \mathcal{O}_i^p L_0]^{-1}$ will have constant inertia for all $i$, if, and only if, the matrices $D_i$ have constant inertia for all $i$.

Now consider the matrix

$$T = \begin{bmatrix} D_i & D_i L_i^* H^* \\ H L_i D_i & R_{e,i} \end{bmatrix}.$$

Two different (block lower-upper and block upper-lower) triangular factorizations of $T$ show that the matrices

$$\begin{bmatrix} D_i & 0 \\ 0 & \underbrace{R_{e,i} - H L_i D_i L_i^* H^*}_{R_{e,i}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \underbrace{D_i - D_i L_i^* H R_{e,i}^{-1} H^* L_i^* D_i}_{D_{i+1}} & 0 \\ 0 & R_{e,i} \end{bmatrix},$$

are congruent and have the same inertia. Thus $D_{i+1}$ will have the same inertia as $D_i$ if, and only if, $R_{e,i}$ has the same inertia as $R_{e,i}$. Therefore we conclude that the $D_i$ have constant inertia for all $i \geq 0$ if, and only if, the $R_{e,i}$ have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$. This establishes the equivalence of (a) and (b).

■

## 8.5 Convergence of the Riccati Recursion with Zero Initial Condition

From the discussions of the prior section we conclude that to verify the convergence of the Riccati recursion for some given initial condition, $P_0$, we need to check the conditions of Theorem 8.4.1. In this section we shall do this for the special case $P_0 = 0$. The motivation for doing so is not so much to illustrate an example of the application of Theorem 8.4.1, but rather because understanding the convergence of the Riccati recursion for $P_0 = 0$ will greatly assist us in obtaining more explicit convergence results for arbitrary initial conditions, $P_0$. This matter will be taken up in Sec. 8.6.

### 8.5.1 The Case of $R > 0$ and $Q - SR^{-1}S^* > 0$

Recall from the discussion at the beginning of Sec. 7.5.1 that there is no loss of generality in assuming $Q - SR^{-1}S^* > 0$ as opposed to $Q - SR^{-1}S^* \geq 0$.

Now the result given below on the convergence of $P_i^0$, the zero-initial-condition solution to the Riccati recursion, is wellknown (see *e.g.*, [Wil71b, Kuc72, Won68a, BRC72]). The standard proof of convergence involves showing that the sequence $\{P_i^0\}$ is nondecreasing and, given the detectability condition, bounded from above. Here, however, we shall give a different proof that uses Theorem 8.4.1.

**Theorem 8.5.1 (Convergence of Riccati Recursion with $P_0 = 0$)** *Consider the Riccati recursion with zero initial condition*

$$P_{i+1}^0 = FP_i^0F^* + GQG^* - (FP_i^0H^* + GS)(R + HP_i^0H^*)^{-1}(FP_i^0H^* + GS)^*, \quad P_0^0 = 0, \tag{8.5.1}$$

*and suppose that $\{F, H\}$ is detectable, $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable, and*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} > 0.$$

*Then $P_i^0$ converges to $P$, the unique stabilizing and positive semi-definite solution of the discrete-time algebraic Riccati equation*

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

**Proof:** We first remark that, from Theorem 7.5.1, since $\{F, H\}$ is detectable and $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable, the DARE has a unique stabilizing and positive semi-definite solution, $P$. Now to prove the Theorem we shall show that $P_i^0 \geq 0$, for all $i \geq 0$. Since $R > 0$, this will imply that

$$R_{e,i}^0 = R + HP_i^0 H^* > 0, \qquad \forall i,$$

*i.e.*, that $R_{e,i}$ has constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$. Therefore using Theorem 8.4.1, the sequence $\{P_i^0\}$ must converge to $P$.

To prove that $P_i^0 \geq 0$, for all $i \geq 0$, we proceed by induction. The case $i = 0$ is obvious since $P_0^0 = 0 \geq 0$. So let us assume that $P_i^0 \geq 0$, for some given $i$. We then may write the Riccati recursion as

$$P_{i+1}^0 = F_{p,i}^0 P_i F_{p,i}^{0*} + \begin{bmatrix} G & -K_{p,i}^0 \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} G^* \\ -K_{p,i}^{0*} \end{bmatrix}.$$

Now since $P_i \geq 0$ and $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} > 0$, the above expression readily shows that $P_{i+1}^0 \geq 0$. This finishes our induction and thus proves the Theorem.

■

### 8.5.2 The Case of Positive $Q - SR^{-1}S^*$

As mentioned at the beginning of Sec. 7.5.3, there is no loss of generality in assuming $S = 0$ and that the (possibly) indefinite matrix $R$ has the form

$$R = \begin{bmatrix} I_{p1} & 0 \\ 0 & -I_{p2} \end{bmatrix}.$$

Therefore, in this case our assumption will reduce to $Q > 0$. Moreover, we shall partition the matrix $H$ according to the partitioning of $R$ and write

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

In this case, as in the positive case studied in the previous section, our approach will be to find conditions for which the matrices

$$R_{e,i}^0 = R + HP_i^0 H^*,$$

have constant inertia for all $i \geq 0$, where $P_i^0$ is the zero-initial-condition solution to the Riccati recursion (8.1.1).

Now since $R_{e,0}^0 = R$ and $R_{e,\infty}^0 = R_e$ (assuming we have convergence), then a necessary condition for $R_{e,i}^0$ to have constant inertia for all $i \geq 0$, is that $R$ and $R_e$ have the same inertia. The surprising result, given below, is that this is sufficient as well. More precisely, if a stabilizing solution to the DARE with $Q > 0$ exists, then if $R$ and $R_e$ have the same inertia, the $R_{e,i}^0$ will have constant inertia for all $i$.

**Theorem 8.5.2 (Zero-Initial-Condition Convergence for $Q > 0$)** *Consider the Riccati recursion with zero initial condition*

$$P_{i+1}^0 = FP_i^0F^* + GQG^* - FP_i^0H^*(R + HP_i^0H^*)^{-1}HFP_i^0F^*, \quad P_0^0 = 0, \qquad (8.5.2)$$

*where*

$$R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},$$

*$\{F, H_1\}$ is detectable, $\{F, GQ^{1/2}\}$ is stabilizable and $Q > 0$. Suppose, moreover, that the DARE*

$$P = FPF^* + GQG^* - FPH^*(R + HPH^*)^{-1}HPF^*, \qquad (8.5.3)$$

*has a stabilizing solution, $P$. Then if the matrices*

$$R \quad \text{and} \quad R_e = R + HPH^*$$

*have the same inertia, the sequence of matrices $\{P_i^0\}$ will converge to $P$.*

**Proof:** The main effort of this proof is to show, under the hypotheses of the Theorem, that the matrices $R_{e,i}^0$ have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$. Once this fact is established, the convergence of $P_i^0$ to $P$ follows immediately from Theorem 8.4.1.

To show that the $R_{e,i}^0$ have constant inertia we shall begin with what may appear as a digression. To this end, consider the time-invariant state-space model

$$\left\{\begin{array}{rcl} x_{i+1} & = & Fx_i + GQ^{1/2}u_i \\ y_i & = & H_1 x_i + v_i \end{array}\right. \tag{8.5.4}$$

and suppose we would like to estimate the linear combination of the states,

$$s_i = H_2 x_i, \tag{8.5.5}$$

using the observations $\{y_j\}_{j=-\infty}^{i}$. For this purpose, let us construct the estimator

$$\left\{\begin{array}{rcl} \hat{x}_{i+1} & = & F\hat{x}_i + K_1(y_i - H_1\hat{x}_i) \\ \hat{s}_i & = & H_2\hat{x}_i + H_2PH_1^*(I_{p_1} + H_1PH_1^*)^{-1}(y_i - H_1\hat{x}_i) \end{array}\right. \tag{8.5.6}$$

where $P$ is the stabilizing solution to the DARE (8.5.3) and $K_1 = FPH_1^*(I_{p_1} + H_1PH_1^*)^{-1}$. It is now straightforward to write down a state-space model for the estimation errors $\tilde{s}_i = s_i - \hat{s}_i$ using (8.5.4) and (8.5.6), as follows

$$\left\{\begin{array}{rcl} \tilde{x}_{i+1} & = & (F - K_1H)\tilde{x}_i + GQ^{1/2}u_i - K_1v_i \\ \tilde{s}_i & = & H_2P(I + PH_1^*H_1)^{-1}(F - K_1H)\tilde{x}_i - H_2PH_1^*(I_{p_1} + H_1PH_1^*)^{-1}v_i \end{array}\right. \tag{8.5.7}$$

where we have defined $\tilde{x}_i = x_i - \hat{x}_i$. Taking $z$-transforms, we may write the above as

$$\tilde{s}(z) = \left[\begin{array}{cc} M_1(z) & M_2(z) \end{array}\right] \left[\begin{array}{c} u(z) \\ v(z) \end{array}\right], \tag{8.5.8}$$

where

$$M_1(z) = H_2(I + PH_1^*H_1)^{-1}(zI - F + K_1H_1)^{-1}GQ^{1/2}$$

and

$$M_2(z) = -H_2PH_1^*(I_{p_1} + H_1PH_1^*)^{-1} - H_2(I + PH_1^*H_1)^{-1}(zI - F + K_1H_1)^{-1}K_1.$$

But from Theorem 7.5.3, since the stabilizing solution to the DARE is such that $R$ and $R_e$ have the same inertia, the transfer matrix $M(z)$ is strictly contractive on the unit circle, $|z| = 1$, *i.e.*,

$$M(z)M^*(z^{-*}) < 1, \qquad \forall |z| = 1.$$

This implies that the mapping from the inputs $\{u_j, v_j\}_{j=-\infty}^{\infty}$ to the estimation error $\{\tilde{z}_j\}_{j=-\infty}^{\infty}$ is strictly contractive. To be more precise, the time-domain representation of this fact is that

$$\sup_{u,v \in l^2} \frac{\|\tilde{s}\|_2^2}{\|u\|_2^2 + \|v\|_2^2} < 1, \tag{8.5.9}$$

where the notation $\|\cdot\|_2$ denotes the 2-norm of a sequence

$$\|a\|_2 = \sum_{j=-\infty}^{\infty} |a_i|_2^2,$$

and $l^2$ is the space of all square-summable sequences.

Let us now fix a constant $i > 0$, and consider the space of those inputs $\{u_j\}$ and $\{v_j\}$ that are zero for all $j > i$. For this class of inputs we may write (8.5.9) as

$$\frac{\sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2}{\sum_{j=-\infty}^{i} |u_j|^2 + \sum_{j=-\infty}^{i} |v_j|^2} < 1,$$

or

$$\frac{\sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2}{\sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1. \tag{8.5.10}$$

We now fix a value for $x_0$, the state vector at time zero, and choose the negative part of the inputs, *i.e.*, $\{u_j^m, v_j^m\}_{j=-\infty}^{-1}$, such that

$$\{u_j^m, v_j^m\}_{j=-\infty}^{-1} = \text{argmin} \left[ \sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 \right],$$

where the $\{u_j, v_j\}$ satisfy the state-space constraints

$$x_{i+1} = Fx_i + GQ^{1/2}u_i, \qquad -\infty < i < 0$$

and generate the given state vector $x_0$. The solution to the above minimization problem is wellknown in LQR control (see *e.g.*, [AM79, KS72]). For our purposes, it suffices to know that the value of the LQ (linear quadratic) cost function at its minimum is given by

$$\sum_{j=-\infty}^{-1} |u_j^m|^2 + \sum_{j=-\infty}^{-1} |v_j^m|^2 = \min_{\{u_j,v_j\}} \left[ \sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 \right] = x_0^* \Pi^{-1} x_0,$$

where $\Pi$ is the unique positive semidefinite solution to the Lyapunov equation[1]

$$\Pi = F \Pi F^* + GQG^*. \tag{8.5.11}$$

With this particular choice of inputs, we may write (8.5.10) as

$$\frac{\sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2}{x_0^* \Pi^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1,$$

or, after discarding the portions of the numerator that involve $j < 0$ and $j > i$,

$$\frac{\sum_{j=0}^{i} |\tilde{s}_j|^2}{x_0^* \Pi^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1. \tag{8.5.12}$$

Note that in the above inequality the $\{x_0, \{u_j\}_{j=0}^{i}, \{v_j\}_{j=0}^{i}\}$ are arbitrary. Moreover, it also readily follows that in (8.5.12) we may replace $\Pi$ with any matrix $P_0^{-1}$, such that

$$P_0^{-1} > \Pi \geq 0, \tag{8.5.13}$$

and write

$$\frac{\sum_{j=0}^{i} |\tilde{s}_j|^2}{x_0^* P_0^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1. \tag{8.5.14}$$

Now it is wellknown in $H^\infty$ theory that if the above inequality holds for all $\{x_0, \{u_j, v_j\}_{j=0}^{i}\}$, then the matrices

$$R_{e,j} = R + HP_jH^*,$$

will have constant inertia for all $0 \leq j \leq i$, where $P_j$ satisfies the Riccati recursion

$$P_{j+1} = FP_jF^* + GQG^* - FP_jH^*(R + HP_jH^*)^{-1}HP_jF^*, \quad P_0.$$

(See Chapter 3. For completeness, a proof of this fact is also given in the Appendix.) Now since our choice of the integer, $i > 0$, was arbitrary we can conclude that the

[1] We have, for simplicity, assumed that $\Pi$ is invertible. The end result does not need this assumption.

$R_{e,j}$ must have constant inertia for *all* $j \geq 0$. Finally, it is straightforward to see that one choice of $P_0$ that satisfies (8.5.13) is $P_0 = 0$. With this choice of initial condition, we readily see that the matrices $R^0_{e,j}$ must have constant inertia for all $j \geq 0$, thus finishing the proof of the Theorem.

■

## 8.6 Convergence of the Riccati Recursion with Arbitrary $P_0$

We shall now study the problem of convergence of the Riccati recursion for the case of an arbitrary initial covariance $P_0$. To do so, instead of comparing, $P_i$, the solution to the Riccati recursion with arbitrary given initial condition, to $P$, the stabilizing solution of the DARE, which resulted in Theorem 8.4.1, we shall compare $P_i$ with the (converging) zero-initial-condition solution, $P^0_i$. The reason for this, as we shall presently see, is that it allows one to obtain more explicit, and easier to verify, conditions on $P_0$ for convergence than those that were obtained in Theorem 8.4.1.

Since we have already proven the convergence of $P^0_i$, the solution to the Riccati recursion (8.1.1) with zero initial condition, we can use the result of Lemma 8.3.2 to compare $P^0_i$ to $P_i$, the solution of the Riccati recursion to an arbitrary initial condition $P_0 = \Pi_0$. In this case we may write

$$P_{i+1} - P^0_{i+1} = \Phi^0_p(i+1,0)\left[I + \Pi_0 \mathcal{O}^0_i\right]^{-1} \Pi_0 \Phi^0_p(i+1,0)^*, \tag{8.6.1}$$

where $\Phi^0_p(i+1,0)$ is the state transition matrix for $F^0_{p,i} = F - K^0_{p,i}H$, and $\mathcal{O}^0_i$ is the observability Gramian for $\{F^0_{p,j}, (R^0_{e,j})^{-1/2}H\}$. Eq. (8.6.1) suggests that we need to study the structure of $\Phi^0_p(i+1,0)$ and $\mathcal{O}^0_i$ in more detail.

In order to do so, we shall prove a series of identities concerning the Riccati recursion and its so-called dual Riccati recursion. Although these identities will be useful for convergence studies, they are also of independent interest since they shed further light on the properties of the Riccati recursion itself.

### 8.6.1 The Dual Riccati Recursion

In this section, for simplicity, we shall assume that $S = 0$. In this case, the Riccati recursion (8.1.1) can be simply written as

$$P_{i+1} = FP_iF^* + GQG^* - FP_iH^*(R + HP_iH^*)^{-1}HP_iF^*. \tag{8.6.2}$$

If $S \neq 0$, we can always write the Riccati recursion as

$$\begin{aligned} P_{i+1} &= (F - GSR^{-1}H)P_i(F - GSR^{-1}H)^* + G(Q - SR^{-1}S^*)G^* - \\ &\quad (F - GSR^{-1}H)P_iH^*(R + HP_iH^*)^{-1}HP(F - GSR^{-1}H)^*, \end{aligned}$$

The above expression implies that by making the transformations

$$F \;\to\; F - GSR^{-1}H \;\;,\;\; Q \;\to\; Q - SR^{-1}S^* \;\;\text{and}\;\; R \;\to\; R$$

we can always reduce the Riccati recursion to one with $S = 0$. Thus, there is no loss of generality in making this assumption.

To present some identities concerning $\Phi_p^0(i+1, 0)$ and $\mathcal{O}_i^0$, we now define $P_i^{a,0}$ as the solution to the so-called *adjoint* or *dual* Riccati recursion with zero initial condition, *i.e.*,

$$P_{i+1}^{a,0} = F^*P_i^{a,0}F + H^*R^{-1}H - F^*P_i^{a,0}G(Q^{-1} + G^*P_i^{a,0}G)^{-1}G^*P_i^{a,0}F, \quad P0^{a,0} = 0. \tag{8.6.3}$$

Note that the dual Riccati is obtained by applying the transformations

$$F \to F^* \;\;,\;\; G \to H^* \;\;,\;\; H \to G^* \;\;,\;\; Q \to R^{-1} \;\;\text{and}\;\; R \to Q^{-1},$$

to the original Riccati (8.6.2).

We can now give our first identity. This identity is known, and can be motivated, in scattering theory [FKL76].

**Lemma 8.6.1 (State Transition Matrix for Dual Riccati)** *Denote by*

$$\Phi_p^{a,0}(i, 0) = \begin{cases} \prod_{j=0}^{i-1}(F^* - K_{p,i}^{a,0}G^*) & i > 0 \\ I & i = 0 \end{cases}, \tag{8.6.4}$$

*the state transition matrix of* $F^* - K^{a,0}_{p,i} G^*$. *Then we have*

$$\Phi^0_p(i,0)^* = \Phi^{a,0}_p(i,0). \tag{8.6.5}$$

We shall defer the proof of Lemma 8.6.1 to the Appendix. For the time being, let us note an important consequence of Lemma 8.6.1 that relates the state transition matrices $\Phi^0_p(i,0)$ (corresponding to the zero-initial-condition solution of the Riccati recursion) and $\Phi_p(i,0) = (F - K_p H)^{i-1}$ (corresponding to any solution of the DARE).

**Lemma 8.6.2 (Relation Between $\Phi^0_p(i,0)$ and $\Phi_p(i,0) = (F - K_p H)^{i-1}$)** *We have the following identity*

$$\Phi^0_p(i,0) = (F - K_p H)^i \left[ I + P P^{a,0}_i \right], \tag{8.6.6}$$

*where*

$$\Phi^0_p(i,0) = (F - K^0_{p,i-1} H) \ldots (F - K^0_{p,1} H) F$$

*is the state transition matrix of* $F - K^0_{p,j} H$ *corresponding to* $P^0_j$, *the zero initial condition solution to the Riccati recursion (8.1.1),* $P$ *is any solution of the DARE (7.2.1), and* $P^{a,0}_i$ *is the zero initial condition solution to the dual Riccati recursion*

$$P^{a,0}_{i+1} = F^* P^{a,0}_i F + H^* R^{-1} H - F^* P^{a,0}_i G (Q^{-1} + G^* P^{a,0}_i G)^{-1} G^* P^{a,0}_i F, \quad P^{a,0}_0 = 0.$$

**Proof:** In this proof we assume that $S = 0$. As mentioned earlier, the results readily extend to the case $S \neq 0$. Let us first note the identity

$$F - K_p H = F - F P H^* (R + H P H^*)^{-1} = F (I + P H^* R^{-1} H)^{-1}. \tag{8.6.7}$$

We now proceed to prove (8.6.6) by induction. For $i = 1$ we have

$$\Phi^0_p(1,0) = F - K^0_{p,0} H = F = \underbrace{F(I + P H^* R^{-1} H)^{-1}}_{F - K_p H} (I + P \underbrace{H^* R^{-1} H}_{P^{a,0}_1}) = (F - K_p H) \left[ I + P P^{a,0}_1 \right],$$

as desired.

Let us now suppose that (8.6.6) is true for $i$. We then have

$$\begin{aligned}\Phi_p^0(i+1,0)=\Phi_p^{a,0}(i+1,0)^* &= \Phi_p^{a,0}(i,0)^* F_{p,i}^{a,0*} \\ &= \Phi_p^0(i,0) F_{p,i}^{a,0*} \\ &= (F-K_pH)^i(I+PP_i^{a,0})F_{p,i}^{a,0*}, \qquad (8.6.8)\end{aligned}$$

where we have twice made use of (8.6.5). On the other hand, using the dual Riccati recursion and an argument similar to (8.6.7) we have

$$F_{p,i}^{a,0}=F^*(I+P_i^{a,0}GQG^*)^{-1}.$$

Using the above expression and proceeding with care yields

$$\begin{aligned}(I+PP_i^{a,0})F_{p,i}^{a,0*} &= (I+PP_i^{a,0})(I+GQG^*P_i^{a,0})^{-1}F \\ &= (I+PP_i^{a,0})\left[I+\underbrace{(P-F(I+PH^*R^{-1}H)^{-1}PF^*)}_{GQG^*}P_i^{a,0}\right]^{-1}F \\ &= (I+PP_i^{a,0})(I+PP_i^{a,0}-F(I+PH^*R^{-1}H)^{-1}PF^*P_i^{a,0})^{-1}F \\ &= F+\underbrace{F(I+PH^*R^{-1}H)^{-1}}_{F-K_pH}PF^*P_i^{a,0}(\underbrace{I+PP_i^{a,0}-F(I+PH^*R^{-1}H)^{-1}PF^*P_i^{a,0}}_{I+GQG^*P_i^{a,0}})^{-1}F \\ &= F+(F-K_pH)P\underbrace{F^*P_i^{a,0}(I+GQG^*P_i^{a,0})^{-1}F}_{P_{i+1}^{a,0}-H^*R^{-1}H} \\ &= F+(F-K_pH)P(P_{i+1}^{a,0}-H^*R^{-1}H) \\ &= F+(F-K_pH)PP_{i+1}^{a,0}-\underbrace{F(I+PH^*R^{-1}H)^{-1}PH^*R^{-1}H}_{F-(F-K_pH)} \\ &= (F-K_pH)PP_{i+1}^{a,0}+(F-K_pH) \\ &= (F-K_pH)(I+PP_{i+1}^{a,0}).\end{aligned}$$

If we now use the last of the above expressions in (8.6.8) we obtain

$$\Phi_p^0(i+1,0)=(F-K_pH)^{i+1}(I+PP_{i+1}^{a,0}),$$

which is the desired result.

∎

Our next identity concerns $\mathcal{O}_i^0$.

**Lemma 8.6.3 (Identity for $\mathcal{O}_i^0$)** *The quantity $\mathcal{O}_i^0$, defined in Lemma 8.3.2, can be expressed as*

$$\mathcal{O}_i^0 = P_{i+1}^{a,0}. \tag{8.6.9}$$

**Proof:** We begin by noting the identity

$$\begin{aligned}
H^*(R_{e,j}^0)^{-1}H &= H^*(R + HP_j^0H^*)^{-1}H \\
&= H^*\left[R^{-1} - R^{-1}H(I + P_j^0H^*R^{-1}H)^{-1}P_j^0H^*R^{-1}\right]H \\
&= \left[I - H^*R^{-1}H(I + P_j^0H^*R^{-1}H)^{-1}P_j^0\right]H^*R^{-1}H \\
&= \left[I - H^*(R + HP_j^0H^*)^{-1}HP_j^0\right]H^*R^{-1}H \\
&= \left[F^* - H^*(R + HP_j^0H^*)^{-1}HP_j^0F^*\right]F^{-*}H^*R^{-1}H \\
&= F_{p,i}^{0*}F^{-*}H^*R^{-1}H
\end{aligned}$$

Now plugging the last of the above equalities into (8.3.8), the expression for $\mathcal{O}_i^0$, yields

$$\begin{aligned}
\mathcal{O}_i^0 &= \sum_{j=0}^{i} \Phi_p^0(j,0)^* F_{p,i}^{0*} F^{-*} H^* R^{-1} H \Phi_p^0(j,0) \\
&= \sum_{j=0}^{i} \Phi_p^0(j+1,0)^* F^{-*} H^* R^{-1} H \Phi_p^0(j,0) \\
&= \sum_{j=0}^{i} \Phi_p^{a,0}(j+1,0) F^{-*} H^* R^{-1} H \Phi_p^{a,0}(j,0)^* \\
&= \sum_{j=0}^{i} \Phi_p^{a,0}(j+1,1) H^* R^{-1} H \Phi_p^{a,0}(j,0)^*.
\end{aligned} \tag{8.6.10}$$

On the other hand, using the dual Riccati recursion (8.6.3), we may write

$$\begin{aligned}
P_{i+1}^{a,0} - P_i^{a,0} &= F_{p,i}^{a,0}(P_i^{a,0} - P_{i-1}^{a,0})F_{p,i-1}^{a,0*} \\
&= \Phi_p^{a,0}(i+1,1)\underbrace{(P_1^{a,0} - P_0^{a,0})}_{H^*R^{-1}H}\Phi_p^{a,0}(i,0)^*,
\end{aligned}$$

which implies that

$$P_{i+1}^{a,0} = \sum_{j=0}^{i} \Phi_p^{a,0}(j+1,1) H^* R^{-1} H \Phi_p^{a,0}(j,0)^*.$$

Comparing this last expression with (8.6.10) yields the desired result.

∎

## 8.7 Conditions on $P_0$ for Convergence

We now have most of the ingredients necessary to study the convergence of the solution of the Riccati recursion for an arbitrary initial condition $P_0 = \Pi_0$, by comparing it with the zero-initial condition solution, $P_i^0$. However, in order to obtain explicit conditions on $P_0$ for convergence, as opposed to the generic conditions of Theorem 8.4.1, we need to pay the price of specializing to the positive case, or the case where $Q - SR^{-1}S^* > 0$. Although these may seem as restrictions, we should remark that they are very important special cases and include the classes of Riccati recursions encountered in both least-squares and $H^\infty$ estimation and control.

### 8.7.1 The Case Positive Case: $R > 0$ and $Q - SR^{-1}S^* > 0$

As frequently mentioned earlier, we may take $S = 0$, so that our assumption becomes $R > 0$ and $Q > 0$. In this case, we can make further assertions regarding $P_i^{a,0}$ the zero-initial-condition solution to the dual Riccati recursion.

**Lemma 8.7.1 (Monotonicity and Boundedness of $P_i^{a,0}$)** *Consider the zero-initial-condition dual Riccati recursion*

$$P_{i+1}^{a,0} = F^* P_i^{a,0} F + H^* R^{-1} H - F^* P_i^{a,0} G (Q^{-1} + G^* P_i^{a,0} G)^{-1} G^* P_i^{a,0} F, \quad P_0^{a,0} = 0$$

*and suppose that $Q > 0$, $R > 0$ and $\{F, GQ^{1/2}\}$ is stabilizable. Then we have the following results:*

*(a) The $\{P_i^{a,0}\}$ form a nondecreasing sequence of Hermitian matrices.*

*(b) The $\{P_i^{a,0}\}$ are bounded from above,* i.e., *there exists a Hermitian matrix $\Pi$ such that*

$$P_i^{a,0} \leq \Pi,$$

*for all $i \geq 0$.*

**Proof:** To prove the monotonicity we shall proceed by induction. For $i = 0$, we have

$$P_1^{a,0} = H^* R^{-1} H \geq 0 = P_0^{a,0}.$$

Now suppose $P_i^{a,0} \geq P_{i-1}^{a,0}$. We shall show $P_{i+1}^{a,0} \geq P_i^{a,0}$. To this end, let us apply (8.3.2) to the dual Riccati recursion and write

$$P_i^{a,0} - P_{i+1}^{a,0} = F_{p,i+1}^{a,0}\left[(P_{i-1}^{a,0} - P_i^{a,0}) - (P_{i-1}^{a,0} - P_i^{a,0})(R_{e,i}^{a,0})^{-1}(P_{i-1}^{a,0} - P_i^{a,0})\right] F_{p,i+1}^{a,0*}.$$

Now since $P_i^{a,0} \geq P_0^{a,0} = 0$, we have $R_{e,i}^{a,0} = R + HP_i^{a,0}H^* > 0$. Therefore both terms on the RHS of the above expression are negative semi-definite. Thus, we can conclude

$$P_{i+1}^{a,0} \geq P_i^{a,0},$$

which shows that the $\{P_i^{a,0}\}$ form a nondecreasing sequence of Hermitian matrices.

To prove boundedness, we shall use the stabilizability of $\{F, GQ^{1/2}\}$. Since $\{F, GQ^{1/2}\}$ is stabilizable, $\{F^*, Q^{*/2}G^*\}$ is detectable, meaning that we can choose a gain matrix $K$ such that $F^* - KQ^{*/2}G^*$ is stable. With this choice of $K$, we can write the dual Riccati recursion as

$$P_{i+1}^{a,0} = (F^*-KQ^{*/2}G^*)P_i^{a,0}(F^*-KQ^{*/2}G^*)^*+H^*R^{-1}H+K^*K-(K_{p,i}^{a,0}-KQ^{*/2})R_{e,i}^{a,0}(K_{p,i}^{a,0}-KQ^{*/2})^*.$$

Since $F^*-KQ^{*/2}G^*$ is stable, and $H^*R^{-1}H+K^*K \geq 0$, there exists a unique positive semi-definite solution, $\Pi$, to the Lyapunov equation.

$$\Pi = (F^* - KQ^{*/2}G^*)\Pi(F^* - KQ^{*/2}G^*)^* + H^*R^{-1}H + K^*K.$$

We shall now prove by induction that $\Pi$ is an upper bound on the sequence $\{P_i^{a,0}\}$. The case $i = 0$ is trivial since

$$\Pi \geq 0 = P_0^{a,0}.$$

Suppose now that $\Pi \geq P_i^{a,0}$. We shall show that $\Pi \geq P_{i+1}^{a,0}$. Indeed, subtracting the Riccati recursion for $P_i^{a,0}$ from the Lyapunov recursion for $\Pi$, yields

$$\Pi-P_{i+1}^{a,0} = (F^*-KQ^{*/2}G^*)(\Pi-P_i^{a,0})(F^*-KQ^{*/2}G^*)^*+(K_{p,i}^{a,0}-KQ^{*/2})R_{e,i}^{a,0}(K_{p,i}^{a,0}-KQ^{*/2})^*.$$

Now both terms on the RHS of the above expression are positive semi-definite. This implies that

$$\Pi \geq P_{i+1}^{a,0},$$

which is our desired result.

■

We can now make some further claims regarding the asymptotic behaviour of the state transition matrix $\Phi_p^0(i,0)$.

**Lemma 8.7.2 (Convergence of $\Phi_p^0(i,0)$)** *If $\{F,H\}$ is detectable and $\{F,GQ^{1/2}\}$ is stabilizable, then*

$$\lim_{i\to\infty}\Phi_p^0(i,0)=0. \tag{8.7.1}$$

**Proof:** The proof uses the identity

$$\Phi_p^0(i,0)=(F-K_pH)^i\left[I+PP_i^{a,0}\right],$$

established in Lemma 8.6.2. The desired result follows immediately from the fact that the $P_i^{a,0}$ are bounded from above and below,

$$0\le P_i^{a,0}\le\Pi<\infty,\quad \forall i\ge 0,$$

and that $F-K_pH$ is stable so that

$$\lim_{i\to\infty}(F-K_pH)^i=0.$$

■

Let us now return to the identity (8.6.1) and write it as

$$P_{i+1}-P_{i+1}^0=\Phi_p^0(i+1,0)\left[I+\Pi_0P_{i+1}^{a,0}\right]^{-1}\Pi_0\Phi_p^0(i+1,0)^*, \tag{8.7.2}$$

where we have also used $\mathcal{O}_i^0=P_{i+1}^{a,0}$. Now, from Lemma 8.7.2, we know that when $\{F,H\}$ is detectable and $\{F,GQ^{1/2}\}$ is stabilizable $\lim_{i\to\infty}\Phi_p^0(i,0)=0$. Therefore from (8.7.2) we conclude that if the matrix $\left[I+\Pi_0P_{i+1}^{a,0}\right]^{-1}$ is uniformly bounded for all $i$, then

$$\lim_{i\to\infty}P_{i+1}=\lim_{i\to\infty}P_{i+1}^0=P.$$

In other words a sufficient condition for the convergence of $P_i$ to $P$, the unique positive semi-definite solution to the DARE is that the matrix

$$I+\Pi_0P_i^{a,0}, \tag{8.7.3}$$

be uniformly nonsingular for all $i$. We shall therefore continue by studying this matrix in more detail.

Let us first suppose that $\Pi_0 \geq 0$. This is, of course, a natural assumption since in estimation problems $\Pi_0$ is the covariance matrix of the random initial state vector $x_0$, and in control problems it is the inverse of the positive weight matrix in the LQ cost function associated with the initial state. In either case, we can perform the factorization $\Pi_0 = \Pi_0^{1/2}\Pi_0^{*/2}$ and write

$$\begin{aligned} \det(I + \Pi_0 P_i^{a,0}) &= \det(I + \Pi_0^{1/2}\Pi_0^{*/2}P_i^{a,0}) \\ &= \det(I + \Pi_0^{*/2}P_i^{a,0}\Pi_0^{1/2}), \end{aligned} \tag{8.7.4}$$

where we have used the identity $\det(I + AB) = \det(I + BA)$. Now since $P_i^{a,0}$ is the solution to the dual Riccati recursion (8.6.3) with zero initial value, it is a positive semi-definite matrix for all $i$. The same is therefore true of the congruent matrix $\Pi_0^{*/2}P_i^{a,0}\Pi_0^{1/2}$, from which we conclude that $I + \Pi_0^{*/2}P_i^{a,0}\Pi_0^{1/2}$ is positive *definite* for all $i$. Thus, in view of (8.7.4), $I + \Pi_0 P_i^{a,0}$ is uniformly nonsingular for all $i$ and we have the following wellknown result [Wil71b, Kuc72, Won68a, BRC72].

**Lemma 8.7.3 (Convergence for $P_0 \geq 0$)** *Consider the Riccati recursion with positive semi-definite initial condition*

$$P_{i+1} = FP_iF^* + GQG^* - K_{p,i}R_{e,i}K_{p,i}^*, \quad P_0 \geq 0. \tag{8.7.5}$$

*If $Q > 0$, $R > 0$, $\{F, H\}$ is detectable and $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable then $P_i$ converges to the unique positive semi-definite matrix, $P$, that satisfies the discrete-time algebraic Riccati equation*

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

Although the above result is quite satisfying, we still have not fully exploited the condition that (8.7.3) be uniformly nonsingular for all $i \geq 0$. It would therefore be interesting to explore whether it is possible to relax the nonnegativity condition on

$\Pi_0$, while still guaranteeing convergence to $P$. We should note that this is not just academic speculation since it will have certain implications for the numerical stability of the Riccati recursion. Indeed, it will show that even if the Riccati variable $P_i$ loses its positive semi-definiteness (due to say, numerical errors) it may still converge to the solution of the DARE.

When $\Pi_0$ is indefinite it no longer yields a factorization of the form described above. However, we can still factorize it as

$$\Pi_0 = \Pi_0^{1/2} J \Pi_0^{*/2},$$

where $J$ is a *signature* matrix whose diagonal elements represent the inertia (or number of positive and negative eigenvalues) of $\Pi_0$.

We now have the following sequence of equalities:

$$\begin{aligned}
\det(I + \Pi_0 P_i^{a,0}) &= \det(I + \Pi_0^{1/2} J \Pi_0^{*/2} P_i^{a,0}) \\
&= \det(I + J \Pi_0^{*/2} P_i^{a,0} \Pi_0^{1/2}) \\
&= \det(J) \cdot \det(J + \Pi_0^{*/2} P_i^{a,0} \Pi_0^{1/2}), \quad \text{since } J^2 = I.
\end{aligned}$$

Using the last of the above expressions, we see that (8.7.3) is uniformly nonsingular if, and only if, the Hermitian matrices $T_i = J + \Pi_0^{*/2} P_i^{a,0} \Pi_0^{1/2}$ are uniformly nonsingular for all $i \geq 0$. Since the matrices $\Pi_0^{*/2} P_i^{a,0} \Pi_0^{1/2}$ are non-decreasing ($P_{i+1}^{a,0} \geq P_i^{a,0}$), the matrices $T_i$ are also non-decreasing. It also follows that *all* the eigenvalues of the matrices $T_i$ are non-decreasing as well. Therefore, a sufficient condition for the $T_i$ to be uniformly nonsingular for all $i \geq 0$, is that all of the negative eigenvalues of $T_0 = J$ remain bounded above by zero as $i$ increases. This, of course, means that the matrices $T_0 = J$ and the $\{T_i\}$ should have the same inertia for all $i \geq 0$. Moreover, the $\{T_i\}$ will have the same inertia as $J$ for all $i \geq 0$ if, and only if,

$$\lim_{i \to \infty} T_i = T = J + \Pi_0^{*/2} P^a \Pi_0^{1/2}, \tag{8.7.6}$$

has the same inertia as $J$. [Note that $P^a$, the limiting solution of the dual Riccati, exists since we have assumed that $\{F, GQ^{1/2}\}$ is stabilizable.]

We have therefore found a sufficient condition for checking whether (8.7.3) is uniformly nonsingular for all $i \geq 0$, in terms of checking the inertia of a *single* matrix

$T$. To facilitate checking the inertia of $T = J + \Pi_0^{*/2} P^a \Pi_0^{1/2}$, consider the following block matrix

$$\begin{bmatrix} -I & (P^a)^{*/2}\Pi_0^{1/2} \\ \Pi_0^{*/2}(P^a)^{1/2} & J \end{bmatrix}, \tag{8.7.7}$$

where we have defined $P^a = (P^a)^{1/2}(P^a)^{*/2}$. Now two different (lower-upper and upper-lower) block triangular factorizations of the above matrix yield

$$\begin{bmatrix} -I & 0 \\ 0 & T \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -I - (P^a)^{*/2}\Pi_0(P^a)^{1/2} & 0 \\ 0 & J \end{bmatrix}. \tag{8.7.8}$$

Due to Sylvester's law of inertia, the above two congruent matrices must have the same inertia. Therefore the matrices $T$ and $J$ will have the same inertia if, and only if, the matrices $-I$ and $-I - (P^a)^{*/2}\Pi_0(P^a)^{1/2}$ have the same inertia. In other words, if, and only if,

$$I + (P^a)^{*/2}\Pi_0(P^a)^{1/2} > 0.$$

We thus have our desired result.

**Theorem 8.7.1 (Convergence of Riccati Recursion with Indefinite $P_0$)** *Consider the Riccati recursion*

$$P_{i+1} = FP_iF^* + GQG^* - K_{p,i}R_{e,i}K_{p,i}^*, \quad P_0 \tag{8.7.9}$$

*where $R > 0$, $Q - SR^{-1}S^* > 0$, $\{F, H\}$ is detectable and $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable. Suppose, moreover, that the initial condition $P_0$ is such that*

$$I + (P^a)^{*/2}P_0(P^a)^{1/2} > 0, \tag{8.7.10}$$

*where $P^a$ is the unique positive semi-definite solution to the dual Riccati recursion. Then $P_i$ converges to the unique positive semi-definite matrix, $P$, that satisfies the discrete-time algebraic Riccati equation*

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

The above result gives a very general description of the conditions for convergence of the Riccati recursion in the positive case. Note that the detectability and stabilizability conditions are the same as those required for the existence of a unique stabilizing solution to the DARE (see Theorem 7.5.1). It is therefore not possible to weaken these conditions very much. Detectability is a necessary condition since otherwise the solution of the Riccati recursion may diverge to infinity. If stabilizability does not hold, then the DARE will have more than one positive semi-definite solution and clearly if we start the recursion with one of these solutions we will always remain at the same value. Therefore, there cannot be convergence to a unique value.

Moreover, (8.7.10) gives a rather general description of the basin of attraction of the positive semi-definite solution of the Riccati recursion. Under additional conditions (8.7.10) can be made more explicit. The result is given below. (The proof is simple and is omitted.)

**Corollary 8.7.1 (Conditions for Convergence)** *When $\{F, H\}$ is observable, the condition (8.7.10) is equivalent to*

$$P_0 > -(P^a)^{-1}. \tag{8.7.11}$$

*Moreover, in this case we have*

$$-(P^a)^{-1} = P_- \triangleq \textit{the infimum over all solutions to the DARE (8.7.9).} \tag{8.7.12}$$

Before closing this section, we should mention an interesting interpretation of the condition (8.7.10) required for the convergence of the Riccati recursion from a given initial condition $P_0$.

**Corollary 8.7.2 (Interpretation of Condition (8.7.10))** *The condition*

$$I + (P^a)^{*/2} P_0 (P^a)^{1/2} > 0,$$

*sufficient for the convergence of the Riccati recursion from a given initial condition $P_0$, is equivalent to the condition that*

$$R_{e,i} = R + H P_i H^* > 0,$$

*for all $i \geq 0$, where $P_i$ is the solution to the Riccati recursion with initial condition $P_0$.*

**Proof:** The proof is similar to the proof of Theorem 8.4.1, part (b), and is omitted for brevity.

$\blacksquare$

Note that the above result implies that we can check the positivity of the matrices $R_{e,i}$ for all $i \geq 0$, in terms of the positivity of a single matrix $I + (P^a)^{*/2} P_0 (P^a)^{1/2}$. However, more interesting is the stochastic interpretation of this result.

To this end, consider the stochastic state-space model

$$\begin{cases} \mathbf{x}_{i+1} &= F\mathbf{x}_i + G\mathbf{u}_i, \quad i \geq 0 \\ \mathbf{y}_i &= H\mathbf{x}_i + \mathbf{v}_i \end{cases} \tag{8.7.13}$$

where the $\{\mathbf{x}_0, \mathbf{u}_i, \mathbf{v}_i\}$ are zero-mean random variables with Gramian (covariance) matrices

$$\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix} \rangle = E \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix}^* = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q & S \\ 0 & S^* & R \end{bmatrix}. \tag{8.7.14}$$

Then it is wellknown (see *e.g.*, [Kai81]) that the $R_{e,i}$ are the Gramians of the, so-called, *innovations process*, $\mathbf{e}_i$, associated with the state-space model (8.7.13-8.7.14). Therefore, the condition (8.7.10) means that we can guarantee convergence for indefinite (and even negative semi-definite) initial conditions, $P_0 = \Pi_0$, as long as the innovations process associated with the state-space model (8.7.13-8.7.14) behaves as a *true* stochastic process, *i.e.*, as one that has positive definite Gramian (covariance) matrix. Finally, since the innovations $\mathbf{e}_i$ are equivalent to the $\mathbf{y}_i$ (in a stochastic sense — see [Jaz70, AM79, Kai81]), we can guarantee convergence for all $P_0 = \Pi_0$ for which the output of the state-space model (8.7.13-8.7.14) behaves as a true stochastic process.

### 8.7.2 The Case of $Q - SR^{-1}S^* > 0$

The special of case $Q - SR^{-1}S^* > 0$ or, equivalently, of $Q > 0$, (since we shall take $S = 0$) is of interest because it is the case that arises in $H^\infty$ filtering and control. To study the convergence in this case, we can begin with Eq. (8.7.2) which we repeat here for convenience,

$$P_{i+1} - P^0_{i+1} = \Phi^0_p(i+1,0)\left[I + \Pi_0 P^{a,0}_{i+1}\right]^{-1} \Pi_0 \Phi^0_p(i+1,0)^*.$$

Using the generalized square-root of the initial condition $\Pi_0 = \Pi_0^{1/2} J \Pi_0^{*/2}$, where $J$ is a signature matrix, we may write the above equation as

$$P_{i+1} - P^0_{i+1} = \Phi^0_p(i+1,0)\Pi_0^{1/2}\left[J + \Pi_0^{*/2} P^{a,0}_{i+1} \Pi_0^{1/2}\right]^{-1} \Pi_0^{*/2} \Phi^0_p(i+1,0)^*. \qquad (8.7.15)$$

The following result is now immediate.

**Lemma 8.7.4 (Comparison with Zero-Initial-Condition Solution)** *Consider the Riccati recursion*

$$P_{i+1} = FP_iF^* + GQG^* - FP_iH^*(R + HP_iH^*)^{-1}HPF^*, \quad P_0 = \Pi_0$$

*where $Q > 0$ and $\{F, GQ^{1/2}\}$ is stabilizable. Moreover, suppose that the associated DARE has a unique stabilizing solution, $P$, and that the zero-initial-condition solution $P^0_i$ converges to $P$. Now if the matrices*

$$T_i = J + \Pi_0^{*/2} P^{a,0}_{i+1} \Pi_0^{1/2}$$

*are uniformly nonsingular for all $i \geq 0$, where $\Pi_0 = \Pi_0^{1/2} J \Pi_0^{*/2}$, and $P^{a,0}_i$ is the zero-initial-condition solution to the dual Riccati recursion*

$$P^{a,0}_{i+1} = F^*P^{a,0}_iF + H^*R^{-1}H - F^*P^{a,0}_iG(Q^{-1} + G^*P^{a,0}_iG)^{-1}G^*P^{a,0}_iF, \quad P^{a,0}_0 = 0,$$

*then $P_i$ will converge to $P$. Two equivalent sufficient conditions for the $T_i$ to be uniformly nonsingular for all $i \geq 0$ are that*

*(a) The matrices $T_i$ have constant inertia for all $i \geq 0$.*

*(b) The matrices $R_{e,i} = R + HP_iH^*$ have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$.*

**Proof:** To prove the first part of the Theorem we shall prove that

$$\lim_{i\to\infty} \Phi_p^0(i,0) = 0. \tag{8.7.16}$$

Once this is established, the convergence of $P_i$ to $P$ follows from (8.7.15), since $\left[J + \Pi_0^{*/2} P_{i+1}^{a,0} \Pi_0^{1/2}\right]^{-1}$ is bounded and $P_i^0$ converges to $P$.

To prove (8.7.16) we can repeatedly use (8.3.1) to write

$$P_{i+1}^0 - P_i^0 = F_{p,i}^0 (P_i^0 - P_{i-1}^0) F_{p,i-1}^{0*} = \Phi_p^0(i+1,1) P_1^0 \Phi_p^0(i,0)^* = \Phi_p^0(i+1,1) GQG^* \Phi_p^0(i,0)^*,$$

where we have used $P_1^0 = GQG^*$. Now we may write

$$(P_{i+1}^0 - P_i^0) F_{p,i}^{0*} = \Phi_p^0(i+1,1) GQG^* \Phi_p^0(i+1,0)^*,$$

which, using Lemma 8.6.1, is equivalent to

$$(P_{i+1}^0 - P_i^0) F_{p,i}^{0*} = \Phi_p^{a,0}(i+1,1)^* GQG^* \Phi_p^{a,0}(i+1,0).$$

This last expression can in turn be written as

$$F^*(P_{i+1}^0 - P_i^0) F_{p,i}^{0*} = \Phi_p^{a,0}(i+1,0)^* GQG^* \Phi_p^{a,0}(i+1,0),$$

and, using the facts that $\lim_{i\to\infty}(P_{i+1}^0 - P_i^0) = 0$ and $Q > 0$, this implies that we must have

$$\lim_{i\to\infty} Q^{*/2} G^* \Phi_p^{a,0}(i+1,0) = 0. \tag{8.7.17}$$

The quantity $\Phi_p^{a,0}(i,0) GQ^{1/2}$ is readily recognized to be the matrix that maps the initial state to the output for the linear system

$$\left\{ \begin{array}{rcl} x_{i+1} & = & (F^* - K_{p,i}^{a,o} G^*) x_i = (F^* - F^* P_i^{a,0} G (R_{e,i}^{a,0})^{-1} G^*) x_i \\ y_i & = & Q^{*/2} G^* x_i \end{array} \right. . \tag{8.7.18}$$

Therefore, (8.7.17) implies that $y_i$ tends to zero for all initial state vectors $x_0$. But (8.7.18) can be rewritten as

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F^* x_i - F^* P_i^{a,0} G (R_{e,i}^{a,0})^{-1} Q^{-*/2} y_i \\ y_i & = & Q^{*/2} G^* x_i \end{array} \right. .$$

and since $\{F^*, Q^{*/2}G^*\}$ is detectable ($\{F, GQ^{1/2}\}$ is stabilizable), $y_i$ cannot tend to zero unless $x_i$ tends to zero as well. Hence, the state transition matrix

$$\Phi_p^{a,0}(i,0) = \Phi_p^0(i,0)^*,$$

tends to zero as $i \to \infty$. (This is just a reformulation of the fact that output feedback cannot destroy detectability — see *e.g.*, [Kai80].)

To prove the second part, we note that if the $T_i$ have constant inertia (equal to the inertia of $J$) for all $i \geq 0$, then they will be uniformly nonsingular as well. Finally, the equivalence of the sufficient conditions (a) and (b) is the same as the proof for Theorem 8.4.1.

■

**Remarks:**

(i) Note that in the above Lemma we still require to check the uniform nonsingularity, or constant inertia, of the sequence of matrices $T_i = J + \Pi_0^{*/2} P_{i+1}^{a,0} \Pi_0^{1/2}$. We saw in the positive case that we could replace checking the inertia of the above infinite sequence of matrices with checking the inertia of a single matrix. This was possible because we were able to show that in the positive case the sequence of matrices $P_i^{a,0}$ were nondecreasing. Since in the case where $Q > 0$, it is not generally true that the $P_i^{a,0}$ are nondecreasing, it does not seem that we can replace the checking of the inertia of the sequence of matrices $\{T_i\}$ with that of a single matrix. Indeed we have not been able to do so.

(ii) The sufficient condition (b) in Lemma 8.7.4 has an interesting interpretation since it is the exact condition that is required for the existence of $H^\infty$ filters or full information controllers (see *e.g.*, [GL95, HSK96b]). Therefore, if a time-varying $H^\infty$ filter, or full information controller, exists for all times $i \geq 0$, then the corresponding Riccati recursion converges, meaning that the time-varying $H^\infty$ filter, or controller, will converge to a time-invariant one.

Despite the result of the preceding Lemma, it is still possible to give some simple description of the basin of attraction of the Riccati recursion when $Q > 0$. However, to do so we need to take a slightly different line of attack. The approach used, follows the one presented in the proof of Theorem 8.5.2, and the result is given below.

**Theorem 8.7.2 (Conditions on $P_0$ for Convergence in the $Q > 0$ Case)** *Consider the Riccati recursion*

$$P_{i+1} = FP_iF^* + GQG^* - FP_iH^*(R + HP_iH^*)^{-1}HFP_iF^*, \quad P_0, \tag{8.7.19}$$

*where*

$$R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix} \quad , \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},$$

*$\{F, H_1\}$ is detectable, $\{F, GQ^{1/2}\}$ is stabilizable and $Q > 0$. Suppose, moreover, that the DARE*

$$P = FPF^* + GQG^* - FPH^*(R + HPH^*)^{-1}HPF^*, \tag{8.7.20}$$

*has a stabilizing solution, $P$, such that the matrices $R$ and $R_e = R + HPH^*$ have the same inertia. Then if the initial condition, $P_0$, is such that*

$$0 \leq P_0 \leq \Pi,$$

*where $\Pi$ is the unique positive (semi)definite solution of the Lyapunov equation,*

$$\Pi = F\Pi F^* + GQG^*,$$

*the sequence of matrices $\{P_i\}$ will converge to $P$.*

**Proof:** Note that due to Theorem 7.5.3, since $P$ is the stabilizing solution to the DARE it is positive semi-definite. Now since $R$ and $R_e$ have the same inertia, Theorem 7.5.3 says that the matrix $F - K_1H$ is stable, where

$$K_1 = FPH_1^*(I_{p_1} + H_1PH_1^*)^{-1}.$$

Now if we assume that $\{F, GQ^{1/2}\}$ is controllable, then the Lyapunov equation, $\Pi = F\Pi F^* + GQG^*$, equation implies that $\Pi > 0$. [We should remark that the controllability assumption is for simplicity and that the end result does not require this assumption.]

Moreover, from the proof of Theorem 8.5.2 for all times $i \geq 0$, we may write

$$\frac{\sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2}{\sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1, \tag{8.7.21}$$

where $\tilde{s}_i = s_i - \hat{s}_i$ is the estimation error associated with estimating $s_i$ (using some estimator that will not concern us here) from the observations $\{y_j\}_{j=-\infty}^{i}$, and where $s_i$ and $y_i$ have the state-space model

$$\left\{ \begin{array}{rcl} x_{i+1} & = & Fx_i + GQ^{1/2} \\ y_i & = & H_1 x_i + v_i \\ s_i & = & H_2 x_i \end{array} \right. \tag{8.7.22}$$

Now for all nonzero sequences $\{u_j\}$ and $\{v_j\}$, we may write (8.7.21) as

$$\left( \sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 - \sum_{j=-\infty}^{-1} |\tilde{s}_j|^2 \right) + \left( \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2 - \sum_{j=0}^{\infty} |\tilde{s}_j|^2 \right) > 0. \tag{8.7.23}$$

Now let us fix an initial state $x_0$ and choose the sequence $\{u_j, v_j\}_{j=-\infty}^{-1}$ such that

$$\{u_j^m, v_j^m\}_{j=-\infty}^{-1} = \text{argmin}_{\{u_j, v_j\}} \left[ \sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 \right], \tag{8.7.24}$$

where the $\{u_j, v_j, \tilde{s}_j\}$ satisfy the state-space constraints

$$x_{i+1} = Fx_i + GQ^{1/2}, \qquad -\infty < i < 0 \tag{8.7.25}$$

and generate the initial state vector $x_0$. The solution to the above minimization problem is wellknown in quadratic game theory and LQR control (see *e.g.*, [AM79, KS72, BO82]). For our purposes it suffices to know that the value of the LQ (linear quadratic) cost function at its minimum is given by

$$\sum_{j=-\infty}^{-1} |u_j^m|^2 + \sum_{j=-\infty}^{-1} |v_j^m|^2 - = \min_{\{u_j, v_j\}} \left[ \sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 - \right] = x_0^* \Pi^{-1} x_0,$$

where $\Pi$ is the solution to the Lyapunov equation, $\Pi = F\Pi F^* + GQG^*$.

With this particular choice of inputs, we may write (8.7.23) as

$$x_0^* \Pi^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2 - \sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2 > 0,$$

or,

$$\frac{\sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2}{x_0^* \Pi^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1,$$

and after discarding that portions of the numerator that involve $j > i$ and $j < 0$,

$$\frac{\sum_{j=0}^{i} |\tilde{s}_j|^2}{x_0^* \Pi^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1. \tag{8.7.26}$$

Note that in the above inequality the $\{x_0, \{u_j\}_{j=0}^{i}, \{v_j\}_{j=0}^{i}\}$ are arbitrary. Moreover, it also readily follows that in the above relation we can replace $P^{-1}$ with any matrix $P_0^{-1}$ such that

$$P_0^{-1} \geq \Pi^{-1},$$

and write

$$\frac{\sum_{j=0}^{i} |\tilde{s}_j|^2}{x_0^* P_0^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1. \tag{8.7.27}$$

Now from Lemma 8.A.1 in the Appendix, we can conclude that the matrices

$$R_{e,j} = R + HP_jH^*,$$

must have constant inertia for all $0 \leq j \leq i$, where $P_j$ satisfies the Riccati recursion (8.7.19). Now since the above is true for all $i \geq 0$, we conclude that with this choice of initial condition the matrices $R_{e,i}$ must have constant inertia for all $i \geq 0$. Therefore, using the result of Theorem 8.4.1, we see that for all initial conditions such that

$$0 \leq P_0 \leq \Pi,$$

the solution to the Riccati recursion, $P_i$, will converge to the unique stabilizing solution, $P$.

■

## 8.8 Conclusion

In this chapter we studied the asymptotic behaviour of the Riccati recursion for time-invariant state-space models. The main result is a general inertia condition for the convergence of the Riccati recursion variable to the unique stabilizing solution of the associated DARE (assuming such a solution exists). We also gave some more simple and more explicit conditions in the special cases encountered in LQR and $H^\infty$ control.

## 8.A Finite-Horizon $H^\infty$ Problems

We begin by proving a Lemma that was used in the proof of Theorem 8.5.2. This Lemma is wellknown in $H^\infty$ theory (see *e.g.*, [GL95, HSK96b] and Chapter 3) and is given below.

**Lemma 8.A.1 (Finite-Horizon $H^\infty$ Filtering)** *Consider the state-space model*

$$\left\{ \begin{array}{rcl} x_{j+1} & = & Fx_j + GQ^{1/2}u_j, \qquad 0 \leq j \leq i \\ y_j & = & H_1 x_j + v_j \end{array} \right.$$

*where the* $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$ *are unknown and the* $\{y_j\}_{j=0}^i$ *are known measurements. Suppose we would like to estimate the linear combination of the states,*

$$s_j = H_2 x_j,$$

*using the observations* $\{y_k\}_{k=0}^j$*, and designate this estimate by* $\hat{s}_j$*. If it is possible to choose the estimates* $\{\hat{s}_j\}_{j=0}^i$ *such that*

$$\frac{\sum_{j=0}^i |s_j - \hat{s}_j|^2}{x_0^* P_0^{-1} x_0 + \sum_{j=0}^i |u_j|^2 + \sum_{j=0}^i |v_j|^2} < 1, \tag{8.A.1}$$

*for all* $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$*, then the matrices*

$$R_{e,j} = R + HP_jH^*,$$

*will have constant inertia for all* $0 \leq j \leq i$*, where* $P_j$ *satisfies the Riccati recursion*

$$P_{j+1} = FP_jF^* + GQG^* - FP_jH^*(R + HP_jH^*)^{-1}HP_jF^*, \quad P_0,$$

*and*

$$R = \left[ \begin{array}{cc} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{array} \right] \quad , \quad H = \left[ \begin{array}{c} H_1 \\ H_2 \end{array} \right].$$

To prove the above Lemma, it will be useful to first establish the following intermediate result.

**Lemma 8.A.2 (Minimization of Indefinite Quadratic Forms)** *Consider the quadratic form*

$$J_i = x_0^* P_0^{-1} x_0 + \sum_{j=0}^{i} u_j^* Q_j^{-1} u_j + \sum_{j=0}^{i} v_j^* R_j^{-1} v_j,$$

*where the* $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$ *satisfy the state-space constraints*

$$\left\{ \begin{array}{rcl} x_{j+1} & = & F x_j + G u_j, \qquad x_0, \quad 0 \le j \le i \\ y_j & = & H x_j + v_j \end{array} \right. \tag{8.A.2}$$

*and where the* $\{P_0, \{Q_j\}_{j=0}^i, \{R_j\}_{j=0}^i\}$ *are (possibly indefinite) Hermitian matrices of the appropriate dimensions. Then, if* $P_0 > 0$ *and* $Q_j > 0$*, for* $j = 0, \ldots i$*, the* $J_i$ *will recursively have minima over the free variables* $\{x_0, \{u_j\}_{j=0}^i\}$ *for all* $i \ge 0$*, if, and only if, the matrices*

$$R \quad \text{and} \quad R_{e,i} = R + H P_i H^* \tag{8.A.3}$$

*have the same inertia for all* $i \ge 0$*, where* $P_i$ *satisfies the Riccati recursion*

$$P_{i+1} = F P_i F^* + G Q G^* - F P_i H^* (R + H P_i H^*)^{-1} H P_i F^*, \quad P_0.$$

**Proof:** To prove the Lemma let us introduce the notation

$$u = \begin{bmatrix} u_0 \\ \vdots \\ u_i \end{bmatrix} \quad , \quad \begin{bmatrix} v_0 \\ \vdots \\ v_i \end{bmatrix} \quad , \quad \begin{bmatrix} y_0 \\ \vdots \\ y_i \end{bmatrix}$$

and

$$Q = Q_0 \oplus \ldots \oplus Q_i \quad , \quad R = R_0 \oplus \ldots \oplus R_i$$

along with the observability and impulse response matrices

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^i \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} 0 & & & \\ HG & 0 & & \\ HFG & HG & & \\ \vdots & \vdots & \ddots & \\ HF^{i-1}G & HF^{i-2}G & \ldots & 0 \end{bmatrix}.$$

With these definitions, we may write the state-space equations (8.A.2) in global form as

$$y = \mathcal{O}x_0 + \Gamma u + v,$$

and the quadratic cost function as

$$\begin{aligned}
J_i &= x_0^* P_0^{-1} x_0 + u^* Q^{-1} u + v^* R^{-1} v \\
&= x_0^* P_0^{-1} x_0 + u^* Q^{-1} u + (\mathcal{O}x_0 + \Gamma u - v) R^{-1} (\mathcal{O}x_0 + \Gamma u - v)^* \\
&= \begin{bmatrix} \begin{bmatrix} x_0^* & u^* \end{bmatrix} & y^* \end{bmatrix} \begin{bmatrix} \begin{bmatrix} P_0^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} + \begin{bmatrix} \mathcal{O} \\ \Gamma \end{bmatrix} R^{-1} \begin{bmatrix} \mathcal{O}^* & \Gamma^* \end{bmatrix} & -\begin{bmatrix} \mathcal{O} \\ \Gamma \end{bmatrix} R^{-1} \\ -R^{-1} \begin{bmatrix} \mathcal{O}^* & \Gamma^* \end{bmatrix} & R^{-1} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} \\ y \end{bmatrix}.
\end{aligned}$$

Using this last expression, we readily see that $J_i$ has a minimum over $\{x_0, u\}$ if, and only if,

$$M = \begin{bmatrix} P_0^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} + \begin{bmatrix} \mathcal{O} \\ \Gamma \end{bmatrix} R^{-1} \begin{bmatrix} \mathcal{O}^* & \Gamma^* \end{bmatrix} > 0.$$

Now consider the matrix

$$T = \begin{bmatrix} \begin{bmatrix} P_0^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} & \begin{bmatrix} \mathcal{O} \\ \Gamma \end{bmatrix} \\ \begin{bmatrix} \mathcal{O}^* & \Gamma^* \end{bmatrix} & -R \end{bmatrix}.$$

Two different (upper-lower and lower-upper) block triangular factorizations of $T$ show that the matrices

$$\begin{bmatrix} M & 0 \\ 0 & -R \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \begin{bmatrix} P_0^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} & 0 \\ 0 & -(\underbrace{R + \mathcal{O}P_0\mathcal{O}^* + \Gamma Q \mathbf{\Gamma}^*}_{R_y}) \end{bmatrix}$$

are congruent and therefore must have the same inertia. But since $\begin{bmatrix} P_0^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} > 0$, we conclude that $M > 0$ if, and only if the matrices $R$ and $R_y$ have the same inertia. Now using the theory of Kalman filtering in Krein spaces (the natural extension of classical Kalman filtering to indefinite metric spaces — see [HSK96c, HSK96b] and Chapter 2) we can perform the (block) triangular factorization of $R_y$ as follows

$$R_y = L R_e L^*,$$

where $L$ is lower triangular with unit diagonal, and

$$R_e = R_{e,0} \oplus \ldots \oplus R_{e,i},$$

with $R_{e,j}$ given by (8.A.3). Thus $M > 0$ (and $J_i$ has a minimum over $\{x_0, u\}$) if, and only if, $R_e$ and $R$ have the same inertia. But since we require that $J_i$ have a minimum over $\{x_0, \{u_j\}_{j=0}^i\}$ for all $i \geq 0$, this means that the block diagonal entries of $R_e$ and $R$ *i.e.*, the matrices $R_{e,i}$ and $R$, must have the same inertia for all $i \geq 0$.

■

**Proof of Lemma 8.A.1:** We first note that if (8.A.1) is true, then for all $i \geq 0$ we must have

$$x_0^* P_0^{-1} + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i} v_j^* v_j - \sum_{j=0}^{i} (\hat{s}_j - s_j)^* (\hat{s}_j - s_j) > 0,$$

for all $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$, subject to the state-space constraints. Now the above quadratic form can be rewritten in the more familiar form

$$x_0^* P_0^{-1} + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i} ( \begin{bmatrix} y_j \\ \hat{s}_j \end{bmatrix} - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} x_j)^* \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix} ( \begin{bmatrix} y_j \\ \hat{s}_j \end{bmatrix} - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} x_j) > 0. \tag{8.A.4}$$

Now if the above quadratic form is to be positive for all $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$ it must have a minimum over the free variables $\{x_0, \{u_j\}_{j=0}^i\}$ (otherwise the $\{x_0, \{u_j\}_{j=0}^i\}$ can be chosen to make the quadratic form arbitrarily negative). By defining

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix}$$

we can readily identify the above quadratic form with the $J_i$ of Lemma 8.A.2. Since the above quadratic form should have a minimum for all $i \geq 0$, we may apply the result of Lemma 8.A.2 to conclude that the matrices

$$R \quad \text{and} \quad R_{e,i}$$

must have the same inertia for all $i \geq 0$, which is the desired result.

■

## 8.B Proof of Lemma 8.6.1

In this section we shall provide a proof for the identity

$$\Phi_p^0(i,0)^* = \Phi_p^{a,0}(i,0). \tag{8.B.1}$$

This identity can be motivated through scattering theory [FKL76]. Here we shall give an algebraic proof that also sheds further light on the Riccati recursion and brings forth certain concepts related to scattering theory (namely, the Hamiltonian matrix).

In what follows we shall assume, without loss of generality, that $S = 0$ (see the discussion at the beginning of Sec. 8.6.1). We shall also assume, for simplicity, that $F$ is invertible, although the final result, *i.e.* (8.B.1), does not require the invertibility of $F$ (see also [FKL76]).

We begin by writing the Riccati recursion as

$$\begin{aligned} P_{i+1} &= FP_iF^* + GQG^* - FP_iH^*(R+HP_iH^*)^{-1}HP_iF^* \\ &= F\left[I - P_iH^*(R+HP_iH^*)^{-1}H\right]P_iF^* + GQG^* \\ &= F(I+P_iH^*R^{-1}H)^{-1}P_iF^* + GQG^*. \end{aligned}$$

The above equation we can write as

$$(I+P_iH^*R^{-1}H)F^{-1}(P_{i+1} - GQG^*) = P_iF^*,$$

or after some simplification

$$F^{-1}P_{i+1} + P_iH^*R^{-1}F^{-1}P_{i+1} - F^{-1}GQG^* - P_iH^*R^{-1}HF^{-1}GQG^* - P_iF^* = 0.$$

Gathering terms yields

$$\begin{bmatrix} I & -P_i \end{bmatrix} \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix} \begin{bmatrix} P_{i+1} \\ I \end{bmatrix}. \tag{8.B.2}$$

The center matrix appearing in the above expression is referred to as the *Hamiltonian* matrix and is denoted by

$$M = \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix}. \tag{8.B.3}$$

It is convenient to expand the block row and column vectors appearing in (8.B.2) into upper triangular matrices, to obtain

$$\begin{bmatrix} I & -P_i \\ 0 & I \end{bmatrix} \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix} \begin{bmatrix} I & P_{i+1} \\ 0 & I \end{bmatrix}.$$

Due to (8.B.2), the $(1,2)$ block entry in the above resulting product is zero. Some simple algebra shows that the remaining block entries are

$$\begin{bmatrix} I & -P_i \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & P_{i+1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} F_{p,i}^{-1} & 0 \\ -H^*R^{-1}HF^{-1} & F_{p,i}^* \end{bmatrix}. \tag{8.B.4}$$

We now consider the product

$$\prod_{j=0}^{i} \begin{bmatrix} I & -P_j \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & P_{j+1} \\ 0 & I \end{bmatrix}.$$

Since

$$\begin{bmatrix} I & -P_j \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & P_j \\ 0 & I \end{bmatrix}^{-1},$$

we readily see

$$\prod_{j=0}^{i} \begin{bmatrix} I & -P_j \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & P_{j+1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -P_0 \\ 0 & I \end{bmatrix} M^{i+1} \begin{bmatrix} I & P_{i+1} \\ 0 & I \end{bmatrix}.$$

On the other hand, using (8.B.4)

$$\prod_{j=0}^{i} \begin{bmatrix} I & -P_j \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & P_{j+1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Phi_p(i+1,0)^{-1} & 0 \\ \times & \times \end{bmatrix},$$

where '$\times$' denotes irrelevant entries.

We thus have the following identity

$$\begin{bmatrix} I & -P_0 \\ 0 & I \end{bmatrix} M^{i+1} \begin{bmatrix} I & P_{i+1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Phi_p(i+1,0)^{-1} & 0 \\ \times & \times \end{bmatrix}.$$

In particular, if we choose $P_0 = 0$, we have

$$M^{i+1} \begin{bmatrix} I & P_{i+1}^0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Phi_p^0(i+1,0)^{-1} & 0 \\ \times & \times \end{bmatrix},$$

which implies

$$\left[M^{i+1}\right]_{11} = \Phi_p^0(i+1,0)^{-1}, \tag{8.B.5}$$

where $[M^{i+1}]_{11}$ denotes the $(1,1)$ block entry of $M^{i+1}$.

In a similar fashion, if one starts with the dual Riccati recursion

$$P_{i+1}^a = F^*P_i^aF + H^*R^{-1}H - F^*P_i^aG(Q^{-1} + G^*P_i^aG)^{-1}G^*P_i^aF,$$

then it turns out that the dual Hamiltonian matrix $M^a$ is simply

$$M^a = M^*.$$

Moreover, the analog of (8.B.5) is

$$\left[(M^a)^{i+1}\right]_{11} = \left[(M^*)^{i+1}\right]_{11} = \left[M^{i+1}\right]_{11}^* = \Phi_p^{a,0}(i+1,0)^{-1}. \tag{8.B.6}$$

Comparing (8.B.5) and (8.B.6) yields

$$\Phi_p^{a,0}(i+1,0) = \Phi_p^0(i+1,0)^*,$$

which is the desired result.

∎

## 8.C Some Global Expressions

In this section we shall give some global expressions for the solution to the Riccati recursion and for various related quantities. Although the material presented here is of interest in its own right, it is also useful in giving alternative derivations of some of the results presented in the chapter that implicitly assumed the invertibility of the state transition matrix, $F$. [We should remark that none of the results given in this chapter require this invertibility assumption. They were made primarily to streamline the arguments that appeared in the main body of the chapter.]

We also remark that the presentation given here is purposefully brief. The active (and interested) reader should be able to fill in the details.

We begin by noting that the standard state-space model can be written in global form as

$$\begin{cases} \mathbf{x}_{i+1} &= \Phi \mathbf{x}_0 + \mathcal{C}\mathbf{u} \ , \\ \mathbf{y} &= \mathcal{O}\mathbf{x}_0 + \Gamma \mathbf{u} + \mathbf{v} \ , \end{cases} \tag{8.C.1}$$

where

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_0 \\ \vdots \\ \mathbf{y}_i \end{bmatrix} \ , \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \vdots \\ \mathbf{u}_i \end{bmatrix} \ , \quad \mathbf{v} = \begin{bmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_i \end{bmatrix} \ , \tag{8.C.2}$$

with

$$\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} \rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & \mathcal{Q} & 0 \\ 0 & 0 & \mathcal{R} \end{bmatrix} \ , \quad \begin{aligned} \mathcal{Q} &= \mathrm{diag}(Q, \ldots Q) \ , \\ \mathcal{R} &= \mathrm{diag}(R, \ldots R) \ , \end{aligned} \tag{8.C.3}$$

and where $\Phi = F^i$ is the state transition matrix,

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{i-1} \end{bmatrix} \quad \text{and} \quad \mathcal{C} = \begin{bmatrix} F^{i-1}G & F^{i-2}G & \ldots & G \end{bmatrix} \ , \tag{8.C.4}$$

are the observability and controllability maps, and

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ HG & 0 & 0 & 0 \\ HFG & HG & 0 & 0 \\ HF^2G & HFG & HG & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{8.C.5}$$

is the impulse response matrix.

Now using the definition of $P_{i+1}$,

$$P_{i+1} = \langle \mathbf{x}_{i+1}, \mathbf{x}_{i+1} \rangle - \langle \mathbf{x}_{i+1}, \mathbf{y} \rangle \langle \mathbf{y}, \mathbf{y} \rangle^{-1} \langle \mathbf{y}, \mathbf{x}_{i+1} \rangle, \tag{8.C.6}$$

it is possible to show the following result.

**Lemma 8.C.1 (Global Expression for $P_i$)** *We have the following identity,*

$$P_{i+1} = \begin{bmatrix} \Phi & \mathcal{C} \end{bmatrix} \left( \begin{bmatrix} \Pi_0^{-1} & 0 \\ 0 & \mathcal{Q}^{-1} \end{bmatrix} + \begin{bmatrix} \mathcal{O}^* \\ \Gamma^* \end{bmatrix} \mathcal{R}^{-1} \begin{bmatrix} \mathcal{O} & \Gamma \end{bmatrix} \right)^{-1} \begin{bmatrix} \Phi^* \\ \mathcal{C}^* \end{bmatrix}. \tag{8.C.7}$$

*In particular, when* $\Pi_0 = 0$,

$$P_{i+1}^0 = \mathcal{C} \left[ \mathcal{Q}^{-1} + \Gamma^* \mathcal{R}^{-1} \Gamma \right]^{-1} \mathcal{C}^*. \tag{8.C.8}$$

Similarly, using the definition $\hat{\mathbf{x}}_{i+1} = \langle \mathbf{x}_{i+1}, \mathbf{y} \rangle \langle \mathbf{y}, \mathbf{y} \rangle^{-1} \mathbf{y}$, we can show that

$$\hat{\mathbf{x}}_{i+1} = (\Phi \Pi_0 \mathcal{O}^* + \mathcal{C} \mathcal{Q} \Gamma^*)(\mathcal{R} + \mathcal{O} \Pi_0 \mathcal{O}^* + \Gamma \mathcal{Q} \Gamma^*)^{-1} (\mathcal{O} \mathbf{x}_0 + \Gamma \mathbf{u} + \mathbf{v}). \tag{8.C.9}$$

Now writing a global expression for $\tilde{\mathbf{x}}_{i+1} = \mathbf{x}_{i+1} - \hat{\mathbf{x}}_{i+1}$ one can deduce that $\Phi_p(i+1, 0)$, the mapping from $\mathbf{x}_0$ to $\tilde{\mathbf{x}}_{i+1}$, is given by the following result.

**Lemma 8.C.2 (Global Expression for $\Phi_p(i+1,0)$)** *We have the following identity,*

$$\Phi_p(i+1, 0) = \Phi - (\Phi \Pi_0 \mathcal{O}^* + \mathcal{C} \mathcal{Q} \Gamma^*)(\mathcal{R} + \mathcal{O} \Pi_0 \mathcal{O}^* + \Gamma \mathcal{Q} \Gamma^*)^{-1} \mathcal{O}. \tag{8.C.10}$$

*In particular, when* $\Pi_0 = 0$,

$$\Phi_p^0(i+1, 0) = \Phi - \mathcal{C} \mathcal{Q} \Gamma^* (\mathcal{R} + \Gamma \mathcal{Q} \Gamma^*)^{-1} \mathcal{O}. \tag{8.C.11}$$

Using a similar (actually dual) argument allows us to write,

$$\Phi_p^{a,0}(i+1,0) = \Phi^* - \mathcal{O}^*\mathcal{R}^{-1}\Gamma(\mathcal{Q}^{-1} + \Gamma^*\mathcal{R}^{-1}\Gamma)^{-1}\mathcal{C}^*. \tag{8.C.12}$$

But the identity,

$$\mathcal{Q}\Gamma^*(\mathcal{R} + \Gamma\mathcal{Q}\Gamma^*)^{-1} = (\mathcal{Q}^{-1} + \Gamma^*\mathcal{R}^{-1}\Gamma)^{-1}\Gamma^*\mathcal{R}^{-1},$$

implies that

$$\Phi_p^{a,0}(i+1,0) = \Phi_p^0(i+1,0)^*, \tag{8.C.13}$$

which is an alternative proof of Lemma 8.6.1 that does not require the invertibility of $F$.

## 8.C.1 Alternative Proof of Lemma 8.6.3

We shall now outline the steps of a (global) alternative proof of the identity

$$P_{i+1}^{a,0} = \sum_{j=0}^{i} \Phi_p^0(j,0)^* H^*(R_{e,j}^0)^{-1} H\Phi_p^0(j,0). \tag{8.C.14}$$

To do so, we begin (with an argument similar to what led to Lemma 8.C.1) to write

$$P_{i+1}^{a,0} = \mathcal{O}^* \left[R + \Gamma Q \Gamma^*\right]^{-1} \mathcal{O}. \tag{8.C.15}$$

In order to write the above result as a sum, we can perform the triangular factorization of the output Gramian matrix

$$R_y = R + \Gamma Q \Gamma^* = L R_e^0 L^*, \tag{8.C.16}$$

with $L$ lower triangular with unit diagonal and $R_e^0$ (block) diagonal. We thus have

$$P_{i+1}^{a,0} = \mathcal{O}^* L^{-*}(R_e^0)^{-1} L^{-1}\mathcal{O}. \tag{8.C.17}$$

Moreover, it is easy to see that the nonzero entries of the $i$-th column of $L$ (where we begin counting the columns from zero) are given by

$$l_i = \begin{bmatrix} I \\ HK_{p,i}^0 \\ HFK_{p,i}^0 \\ HF^2K_{p,i}^0 \\ \vdots \end{bmatrix}. \tag{8.C.18}$$

The readily verified identity,

$$\mathcal{O} = L \begin{bmatrix} H \\ HF_{p,0} \\ HF_{p,1}F_{p,0} \\ \vdots \end{bmatrix}, \tag{8.C.19}$$

now yields the desired result.

### 8.C.2 Alternative Proof of Lemma 8.3.2

We first use Lemma 8.C.1 to write

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \begin{bmatrix} \Phi & \mathcal{C} \end{bmatrix} \left((A+\delta A)^{-1} - A\right) \begin{bmatrix} \Phi^* \\ \mathcal{C}^* \end{bmatrix}, \tag{8.C.20}$$

where we have defined

$$A = \begin{bmatrix} (\Pi_0^{(1)})^{-1} & 0 \\ 0 & \mathcal{Q}^{-1} \end{bmatrix} + \begin{bmatrix} \mathcal{O}^* \\ \Gamma^* \end{bmatrix} \mathcal{R}^{-1} \begin{bmatrix} \mathcal{O} & \Gamma \end{bmatrix} \quad \text{and} \quad \delta A = \begin{bmatrix} (\Pi_0^{(2)})^{-1} - (\Pi_0^{(1)})^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{8.C.21}$$

Now defining the following factorization of $\delta A$,

$$\delta A = \begin{bmatrix} B \\ 0 \end{bmatrix} J \begin{bmatrix} B^* & 0 \end{bmatrix}, \tag{8.C.22}$$

where $B$ is full rank and $J$ is an appropriate inertia matrix, allows us to write

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \begin{bmatrix} \Phi & \mathcal{C} \end{bmatrix} A^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \left( J + \begin{bmatrix} B^* & 0 \end{bmatrix} A^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B^* & 0 \end{bmatrix} A^{-1} \begin{bmatrix} \Phi^* \\ \mathcal{C}^* \end{bmatrix}. \tag{8.C.23}$$

Now using, the readily verified facts that

$$\begin{bmatrix} \Phi & \mathcal{C} \end{bmatrix} A^{-1} = \begin{bmatrix} \Phi_p^{(1)}(i+1,0)\Pi_0^{(1)} & \times \end{bmatrix}, \tag{8.C.24}$$

where $\times$ denotes irrelevant entries, and

$$\begin{bmatrix} B^* & 0 \end{bmatrix} A^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} = B^*(\Pi_0^{(1)} - \Pi_0^{(1)}\mathcal{O}^* R_y^{-1} \mathcal{O} \Pi_0^{(1)})B, \tag{8.C.25}$$

where $R_y \triangleq \mathcal{R} + \mathcal{O}\Pi_0\mathcal{O}^* + \Gamma\mathcal{Q}\Gamma^*$, allows us to write,

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \Phi_p^{(1)}(i+1,0)\Pi_0^{(1)}B\left[J + B^*(\Pi_0^{(1)} - \Pi_0^{(1)}\mathcal{O}^*R_y^{-1}\mathcal{O}\Pi_0^{(1)})B\right]^{-1}B^*\Pi_0^{(1)}\Phi_p^{(1)}(i+1,0)^*,$$

and, upon further simplification,

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \Phi_p^{(1)}(i+1,0)\left[I + (\Pi_0^{(2)} - \Pi_0^{(1)})\mathcal{O}^*R_y^{-1}\mathcal{O}\right]^{-1}(\Pi_0^{(2)} - \Pi_0^{(1)})\Phi_p^{(1)}(i+1,0)^*. \tag{8.C.26}$$

Now an argument similar to what was presented in Sec. 8.C.1 allows us to conclude

$$\mathcal{O}^*R_y^{-1}\mathcal{O} = \sum_{j=0}^{i}\Phi_p^{(1)}(j,0)^*H^*(R_{e,j}^{(1)})^{-1}H\Phi_p^{(1)}(j,0). \tag{8.C.27}$$

But this is the same as the expression given for $\mathcal{O}_i^{(1)}$, given in Lemma 8.3.2, and we have therefore established the identity

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \Phi_p^{(1)}(i+1,0)\left[I + (\Pi_0^{(2)} - \Pi_0^{(1)})\mathcal{O}_i^{(1)}\right]^{-1}(\Pi_0^{(2)} - \Pi_0^{(1)})\Phi_p^{(1)}(i+1,0)^*, \tag{8.C.28}$$

which is our desired result.