A Sub-optimal Algorithm to Synthesize Control Laws for a Network of Dynamic Agents

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Abstract

We study the synthesis problem of an LQR controller when the matrix describing the control law is constrained to lie in a particular vector space. Our motivation is the use of such control laws to stabilize networks of autonomous agents in a decentralized fashion, with the information flow being dictated by the constraints of a pre-specified topology. In this paper, we consider the finite-horizon version of the problem and provide both a computationally intensive optimal solution and a sub-optimal solution that is computationally more tractable. Then we apply the technique to the decentralized vehicle formation control problem. It is numerically illustrated that while the loss in performance due to the use of the sub-optimal solution is not huge, the topology can have a large effect on performance.

1. Introduction and Motivation

Control of dynamic agents coupled to each other through an information flow network has emerged as a topic of major interest in recent years. Such a setting can be used to model many real-life situations, such as air traffi c control, satellite clusters, swarms of robots, UAV formations, and potentially such applications as the Internet. Compared with the more traditional applications of control theory, there are fundamentally new features introduced in this problem. The topology of the information network can have many effects. On one hand, it introduces instability if the information being fed through the network adds on constructively to the disturbance at a node (Fax and Murray 2004); on the other, for cooperative goals, it leads to a better performance than if agents do not share information.

As a result of the above-mentioned properties, this problem has been garnering increasing attention (e.g., see (Antsaklis and Baillieul 2004) and the references therein). A Nyquist-like condition for stability of a formation using the individual plant transfer function and the Laplacian of the graph generated by the topology of the information flow network was obtained in (Fax 2001, Fax and Murray 2004). Coordination of a group of autonomous agents when the graph topology changes over time was considered in (Jadbabaie *et al.* 2003) who presented stability results for the case when the switching rule satisfies certain properties. These results were expanded in (Ren and Beard 2004). A general framework for decentralized analysis of stability of interconnected systems where the topology can potentially be time-varying was presented in (Gupta *et al.* 2003, Saber and Murray 2004).

However, most of the work so far has centered on stability analysis of the formation assuming certain control laws in place. A more general question is that of synthesis of the control law to be used by the agents in such a formation, such that some cost function is optimized. The defining feature of the problem is that while the cost function involves all the individual agents in the formation, the pre-specified topology of the formation imposes constraints on the form of the control law by limiting the information available to various agents at any time. Thus, it is not realistic to assume that an agent would know the state of all the other agents in the formation at any given time and be able to use it to calculate the control input. These features make the problem a decentralized control problem with arbitrary information flow patterns, which is, in general, much harder to solve than the traditional optimal control problem.

Research in decentralized control has a long history. (Witsenhausen 1968, Witsenhausen 1971) showed that under the decentralized information constraints, a linear controller is not optimal in general and also that the cost function need not be convex in the controller variables. A discrete equivalent of Witsenhausen's counter-example was given in (Papadimitriou and Tsitsiklis 1986) where it was also shown that the problem of finding a stabilizing controller under information pattern constraints is NP-complete. For particular information structures, the problem has been solved, e.g., see (Fan *et al.* 1994). Some researchers have also studied this problem under the assumption of spatial invariance by using a multidimensional approach (e.g., see (Bamieh *et al.* 2002, D'Andrea and Dullerud 2003.)). (Rotkowitz and Lall 2002) gave certain invariance conditions under which the problem retains the convex character. (Langbort *et al.* 2004) presented sufficient LMI conditions that can be used for synthesizing a sub-optimal distributed controller. A different approach for solving the problem was inspired by the design of reduced-order controllers, e.g., (Levine *et al.* 1971). This approach was used to obtain numerical algorithms for solving the optimal linear control with arbitrary number of free parameters for the infinite horizon case as in (Soderstrom 1978, Wenk and Knapp 1980). In (Gupta *et al.* 2004), this algorithm was explored for the case of vehicle formations and in particular it was proven that in this case, it was always possible to choose a feasible initial point. A similar algorithm can be applied to the finite horizon problem, as described in (Anderson and Moore 1971), but the computational difficulties were pointed out in (Kleinman *et al.* 1968). The vehicle formation problem was also considered in (de Castro 2003) where the H_2 synthesis problem was posed as an optimization problem and a sub-optimal solution presented. (Bemporad *et al.* 2002) considered the constrained LQR problem and came up with a numerical algorithm for the optimal piecewise affine controller. The algorithm was extended to the case of infinite-time horizon by (Grieder *et al.* 2003). A convex approach towards synthesizing control laws for solving distributed averaging problems was given in (Xiao and Boyd 2004). Receding horizon control for the problem was explored in (Dunbar and Murray 2004, Franco *et al.* 2004). A good survey of the attempts to solve the related fixed order and static output feedback problems can be found in (Syrmos *et al.* 1997, Blondel *et al.* 1995, Darbha *et al.* 2004) and the references therein.

As has been pointed out in many of the above mentioned works, the problem of finding a linear optimal controller that satisfies some arbitrary constraints is very difficult. Even the question of stabilizability through a structured controller is NP-hard, in general (Blondel *et al.* 1995). Unless the problem has some special structure, finding the optimal controller with a prescribed structure has remained an open problem. Even the algorithms for finding sub-optimal controllers tend to involve numerical optimization and hence face convergence problems unless the problem satisfies some conditions (Rotkowitz and Lall 2002). In this paper, we set up the LQR problem for the control of a network of autonomous agents with a given information flow topology.

Even if the dynamics of the agents are not coupled and the only coupling present is due to the cost function, the optimal control law, in general, requires every agent to use knowledge about every other agent. We impose the constraint of a linear control law that satisfies a pre-specified topology in that any agent uses only the information about a pre-specified set of agents with which it can communicate. We solve for the optimal control law for a finite time horizon under these constraints. We see that calculation of the optimal control is computationally expensive and provide a sub-optimal solution instead which is computationally more tractable. The chief contribution of this paper are these algorithms. Since both the algorithms we present involve only solving linear equations, they do not suffer from convergence problems encountered in many existing approaches that utilize, e.g., gradient descent algorithms.

The outline of the paper is as follows. We address a few mathematical preliminaries in the next section. Then we set up and solve the constrained controller synthesis problem. We see that calculating the optimal solution is computationally intensive and hence propose a simpler sub-optimal solution. Then we present examples to illustrate the concepts and the algorithm. We see numerically that the loss in performance by choosing the sub-optimal algorithm is not huge. We end with conclusions and present some avenues for further work.

2. Mathematical Preliminaries and Notations

By a network of interconnected dynamic agents, we mean a system of agents whose dynamics are not coupled but in which every agent can use the information from a prescribed set of other agents (called its out-neighbors) for calculating its control input. The flow of information is thus described by identifying the set of out-neighbors for each agent and is referred to as the information flow topology. Consider a network of n agents. Together with the information flow topology, the network can be represented by a graph in which the agents are vertices and the allowed communication links are edges. We refer to the agents variously as vertices, nodes or vehicles and the network as a graph or a formation.

Consider a graph with n nodes, the vertex set $V(G) = \{v_i\}_{i=1}^{n}$ and the edge set $E(G)$. The *adjacency matrix* (see, e.g., (Biggs 1974) for more details) denoted by A is a square matrix of size $n \times n$, defined as follows

$$A_{ij} = \begin{cases} 1 & v_i v_j \ \epsilon \ E(G) \\ 0 & \text{otherwise.} \end{cases}$$

If we denote the out-degree of node v_i by O_i, then the *degree matrix* denoted by D is defined to be a square matrix of size $n \times n$, such that

$$D_{ij} = \begin{cases} O_i & i = j \\ 0 & \text{otherwise.} \end{cases}$$

We define the *Laplacian* of a graph by the following equation

$$L = D - A.$$

We denote the expectation of a random variable X by $E[X]$. The covariance matrix of a random variable X with zero mean is defined by $E\left[XX'\right]$, where X' is the transpose of matrix X. The covariance matrix is always a positive semi-definite matrix.

The trace of a square matrix X, denoted by trace(X), is defined as the sum of its diagonal elements. It is known that the trace is also the sum of the eigenvalues of X. The trace operator satisfies the following properties (assume X, Y and Z to be compatible matrices; v is a column vector).

1) $\text{trace}(X+Y) = \text{trace}(X) + \text{trace}(Y)$.
2) $\text{trace}(XYZ) = \text{trace}(ZXY)$.
3) $E[v'Wv] = E[\text{trace}(Wvv')]$.

In the last equation if W is a constant matrix, the right hand side can be further rewritten as $\text{trace}(WE[vv'])$.

We denote the transpose of a matrix X by X'. For two matrices A and B, we write $A > B$ if $A - B$ is a positive definite matrix. For a matrix $m \times n$ matrix $X = [x_{ij}]$, the operation $\text{vec}(X)$

results in a $mn \times 1$ column vector with elements

$$\text{vec}(X) = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{m1} \\ x_{12} \\ \vdots \\ x_{mn} \end{bmatrix}.$$

The operation $A \otimes B$ denotes the Kronecker product (also called the direct product) between two matrices A and B (see (Lancaster 1969) for details). It can be shown that for suitably dimensioned matrices A, X and B,

$$\text{vec}(AXB) = (B' \otimes A)\,\text{vec}(X). \tag{1}$$

3. Problem Formulation

Consider a formation of n agents, in which the i-th agent evolves according to the equation

$$x^i_{k+1} = \Phi x^i_k + \Gamma u^i_k + w^i_k,$$

where the control law u^i_k is given by

$$u^i_k = F^{i,1}_k x^i_k + \sum_{\text{all out-neighbors } j} F^{ij,2}_k \left(x^i_k - x^j_k\right).$$

Assume that the noise w^i_k is zero-mean, Gaussian and white. On stacking the state x^i of all the agents, we can obtain the system state vector x, whose evolution is described by

$$\begin{aligned} x_{k+1} &= (I \otimes \Phi)x_k + (I \otimes \Gamma)u_k + w_k \\ u_k &= (\text{diag}(F^{i,1}_k) + L_{\text{gen},k})x_k, \end{aligned} \tag{2}$$

where I is identity matrix of suitable dimensions and $\text{diag}(F^{i,1}_k)$ is a block diagonal matrix with $F^{i,1}_k$'s along the diagonal and zero matrices elsewhere. The vectors u_k and w_k are obtained by stacking the control laws and the noises for the individual agents, respectively. $L_{\text{gen},k}$ is a generalization of the Laplacian matrix of the graph and is formed as follows. Create the adjacency matrix A for the network. Then replace each unity element that is at the (i, j)-th place by $-F_k^{ij,2}$. Replace the diagonal element in the i-th row by a matrix which is the sum of the matrices $F_k^{i1,1}$, $F_k^{i2,1}$, $\cdots$, $F_k^{i(i-1)}$, $F_k^{i(i+1)}$, $\cdots$, $F_k^{in,1}$. Rest of the zero elements are replaced by zero matrices of appropriate dimensions. Note that the topological constraints on the form of control law are inherent in the structure of $L_{\text{gen},k}$.

In this paper we will assume that the topology of the network is known at any time step, although it may be time-varying. We ignore issues such as quantization error and message loss when agents communicate over the links. Note that if all the vehicles are not identical, equations similar to (2) can easily be obtained. The matrices $I \otimes \Phi$ and $I \otimes \Gamma$ will be replaced by block diagonal matrices diag(Φ^i) and diag(Γ^i), but other details remain similar. We begin by discussing the questions of stabilizability and controllability of the formation under a specified topology constraint.

3-1. Stabilizability

Two questions arise immediately:

- Is it possible to stabilize a formation using information from other vehicles when the vehicles are individually not stable. In other words, if a vehicle is unstable, can the formation be stabilized by the exchange of information between different agents?
- Are some topologies inherently unstable in that even if the agents are stable, the information flow will always make it impossible to stabilize the formation?

We note the following result originally presented in (Gupta *et al.* 2004).

Proposition 1. *Consider a formation of interconnected dynamic agents as defined in section 2.*

1) *A formation is controllable if and only if each individual agent is controllable.*
2) *A formation is stabilizable if and only if each individual agent is stabilizable.*

Proof: We use the notation introduced above. Let the matrix Φ be in $\mathbf{R}^m$ and there be n agents in the formation. As can be seen from (2), for controllability of the formation, we want the following matrix to have rank mn (Kwakernaak and Sivan 1972)

$$M_1 = \left[\begin{array}{ccccc} I \otimes \Gamma & (I \otimes \Phi)(I \otimes \Gamma) & (I \otimes \Phi)^2(I \otimes \Gamma) & \cdots & (I \otimes \Phi)^{mn-1}(I \otimes \Gamma) \end{array} \right].$$

Using the standard property of Kronecker product

$$(a \otimes b)(c \otimes d) = ac \otimes bd,$$

we can rewrite M_1 as

$$M_1 = \left[\begin{array}{ccccc} I \otimes \Gamma & (I \otimes \Phi\Gamma) & (I \otimes \Phi^2\Gamma) & \cdots & (I \otimes \Phi^{mn-1}\Gamma) \end{array} \right].$$

This matrix has rank mn if and only if the following matrix has rank m

$$M_2 = \left[\begin{array}{ccccc} \Gamma & \Phi\Gamma & \Phi^2\Gamma & \cdots & \Phi^{mn-1}\Gamma \end{array} \right].$$

Since $\Phi \in \mathbf{R}^m$, the equivalent condition is that the matrix

$$M_3 = \left[\begin{array}{ccccc} \Gamma & \Phi\Gamma & \Phi^2\Gamma & \cdots & \Phi^{m-1}\Gamma \end{array} \right],$$

be rank m. But M_3 being rank m is simply the condition for the individual agent being controllable. Thus the formation is controllable if and only if each individual agent is controllable. This proves the first part. The second part also follows from the above proof. The subspace not spanned by the columns of M_1 is stable if and only if the subspace not spanned by the columns of M_3 is stable. $\square$

3-2. Designing the Control Law

From (2), it can be seen that the problem of designing a control law under the topological constraints is equivalent to solving the control design problem for the system

$$\begin{aligned} x_{k+1} &= (I \otimes \Phi)\, x_k + (I \otimes \Gamma)\, u_k + w_k \\ u_k &= F_k x_k, \end{aligned} \tag{3}$$

with the additional constraint that F_k should have those elements as 0 which correspond to zero entries in the $L_{\text{gen},k}$ of the interconnection topology formed as above. F_k can then readily be cast in the form $\text{diag}(F_k^{i,1}) + L_{\text{gen},k}$ and the matrices $F_k^{i,1}$ and $F_k^{ij,2}$ obtained. It is fairly obvious that constraining the control F_k to have some elements zero forces us to consider only those matrices that live in a particular sub-space of the vector space of all matrices with the same dimensions as F_k. We now define the cost function we are aiming to minimize and solve the problem of finding the optimal control law.

4. The Optimal Constrained Control Law

Denote $A = I \otimes \Phi$ and $B = I \otimes \Gamma$ and rewrite (3) as

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k \\ u_k &= F_k x_k, \end{aligned} \tag{4}$$

where the initial condition x_0 is random and is Gaussian with zero-mean and covariance R_0. The noise w_k is also random white zero-mean Gaussian with covariance R_w. In the above equations, $x_k \in \mathbf{R}^n$ is the state of the system and $u_k \in \mathbf{R}^m$ is the control input. We wish to minimize the cost function

$$J_T = E\left[\sum_{k=0}^{T} \{x_k' Q x_k + u_k' R u_k\}\right] + E\left[x_{T+1}' P_{T+1}^c x_{T+1}\right], \tag{5}$$

where Q and R are positive definite. This is the classical LQR design problem. We can find the optimal control law through solving the discrete-time Riccati recursion. Suppose we now wish to additionally constrain the control law to lie within a space spanned by the basis vectors $\{\Lambda_j$, $j = 1, 2, \ldots, N\}$. Thus the problem is to find a control law of the form

$$F_k = \sum_{j=1}^{N} \alpha_k^j \Lambda^j, \tag{6}$$

where α_k^j's are scalars, that minimizes the cost function (5).

Remarks:

1) It is fairly obvious that the optimal constrained control law would not, in general, be the projection of the optimal control law on to the sub-space we are interested in. This is

reminiscent of the fact that the optimal causal estimate for a random variable is not the same as the projection of the general optimal estimate on to the causal sub-space (Kailath *et al.* 2000).

2) Requiring apriori that the controller be linear might be a non-trivial assumption. But this allows us to derive algorithms for solving the problem and leads to sharper results.

4-1. Preliminary Result

In this subsection we prove an intermediate result that we will use later. First note the following fact.

Lemma 2. *Suppose W is positive semi-definite and $P(K)$ denote any matrix-valued function of the matrix argument K. If $P(K) > P(K_0)$, then trace $(P(K)W) \geq$ trace $(P(K_0)W)$.*

Proof: Since $P(K) > P(K_0)$, we have $P(K) - P(K_0) > 0$. Also W is positive semi-definite, thus $W^{\frac{1}{2}}$ is defined. Hence we note that trace$\left(W^{\frac{T}{2}}\left(P(K) - P(K_0)\right)W^{\frac{1}{2}}\right) \geq 0$ or that trace $\left(\left(P(K) - P(K_0)\right)W\right) \geq 0$. But this means trace $(P(K)W) \geq$ trace $(P(K_0)W)$, which proves the assertion. □

Using this lemma we can prove the following.

Proposition 3. *Consider the cost function*

$$C = E\left[\begin{bmatrix} K_1Y_1 - X_1 \\ K_2Y_2 - X_2 \\ \vdots \\ K_nY_n - X_n \end{bmatrix}' W \begin{bmatrix} K_1Y_1 - X_1 \\ K_2Y_2 - X_2 \\ \vdots \\ K_nY_n - X_n \end{bmatrix}\right],$$

where K_i's are arbitrary matrices while Y_i's and X_i's are vectors of suitable dimensions such that the cost function C is well-defined. Suppose that W can be written in the form

$$W = \begin{bmatrix} W_{1,1} & W_{1,2} & \dots & W_{1,n} \\ W_{2,1} & & \dots & W_{2,n} \\ \vdots & & \ddots & \vdots \\ W_{n,1} & W_{n,2} & \dots & W_{n,n} \end{bmatrix},$$

where the blocks $W_{i,j}$ are of appropriate sizes so that the product $X_i' W_{i,j} X_j$ is well defined and that W is symmetric and positive definite. Then the optimal K_i's minimizing the cost function are given by the solution to the coupled matrix equations

$$K_j = W_{j,j}^{-1} \left[\sum_i W_{j,i} R_{X_i Y_j} - \sum_{i \neq j} W_{j,i} K_i R_{Y_i Y_j} \right] R_{Y_j}^{-1}, \qquad \forall j = 1, 2, \cdots, n,$$

where $R_{Y_i Y_j} = E\left[Y_i Y_j'\right]$ and $R_{X_i Y_j} = E\left[X_i Y_j'\right]$.

Proof: For each j, we can write the terms depending on K_j as

$$C_j = \text{trace}\left(K_j R_{Y_j} K_j' W_{j,j} - K_j \Psi - \Psi' K_j' \right),$$

where

$$\Psi = \left[\sum_i R_{Y_j X_i} W_{i,j} - \sum_{i \neq j} R_{Y_j Y_i} K_i' W_{i,j} \right].$$

Thus K_j needs to be chosen so as to minimize C_j. The minimization is of the form

$$\min_X \text{trace}\left(XAX'B + XC + C'X' \right),$$

where B is invertible and positive defi nite. This can be rewritten as

$$\begin{aligned}
\min_X \text{trace}\left(XAX'B + XC + C'X' \right) &= \min_X \text{trace}\left(XAX'B + XCB^{-1}B + C'X'BB^{-1} \right) \\
&= \min_X \text{trace}\left(XAX'B + XCB^{-1}B + B^{-1}C'X'B \right) \\
&= \min_X \text{trace}\left(\left(XAX' + XCB^{-1} + B^{-1}C'X' \right) B \right).
\end{aligned}$$

Now we use lemma 2. Thus our problem reduces to that of determining X such that $XAX' + XCB^{-1} + B^{-1}C'X'$ is minimized. We complete the squares to write

$$XAX' + XCB^{-1} + B^{-1}C'X' = \left(X + B^{-1}C'A^{-1} \right) A \left(X + B^{-1}C'A^{-1} \right)' - B^{-1}C'A^{-1}C.$$

Thus the minimizing $X = -B^{-1}C'A^{-1}$. Applying this to our original problem of determining

K_j, we see that

$$K_j = W_{j,j}^{-1} \left[\sum_i W_{j,i} R_{X_i Y_j} - \sum_{i \neq j} W_{j,i} K_i R_{Y_i Y_j} \right] R_{Y_j}^{-1}.$$

This completes the proof. $\square$

Note that for calculation of the K_j's, we can use the identity (1). Thus we obtain for each K_j, the equation

$$\text{vec}(K_j) = \text{vec} \left(W_{j,j}^{-1} \sum_i W_{j,i} R_{X_i Y_j} \right) - \sum_{i \neq j} \left[\left(R_{Y_i Y_j} R_{Y_j}^{-1} \right)' \otimes \left(W_{j,j}^{-1} W_{j,i} \right) \text{vec}(K_i) \right].$$

We have one such equation for each K_j, $j = 1, \ldots, n$. These equations can readily be solved to obtain the values of $\text{vec}(K_j)$ and from them the matrices K_j can easily be determined.

4-2. The Optimal Control Law

From (5) we see that the cost function to be minimized is

$$J_T = E \left[\sum_{k=0}^{T} u_k' R u_k + \sum_{k=0}^{T} x_k' Q x_k \right] + E \left[x_{T+1}' P_{T+1}^c x_{T+1} \right].$$

Using the equation

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} + \sum_{j=0}^{k-1} A^j w_{k-1-j}$$

and the fact that the noise w_k is white and zero-mean allows us to rewrite the cost function in the form

$$J_T = E \left[\Gamma' \mathbf{F} \Gamma + \Gamma' \mathbf{G} \Lambda + \Lambda' \mathbf{G}' \Gamma + \Lambda' \mathbf{H} \Lambda \right]. \tag{7}$$

In the above equation

$$\Gamma = \begin{bmatrix} x_0' & w_0' & w_1' & \cdots & w_T' \end{bmatrix}'$$

is the vector of all the random variables involved. Similarly,

$$\Lambda = \begin{bmatrix} u_0' & u_1' & \cdots & u_T' \end{bmatrix}'$$

is the control vector that is the optimization variable, and the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ are functions of A, B, R, Q and P^c_{T+1}. The additional constraint on Λ is that it has to be of the form

$$\Lambda = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_T \end{bmatrix} = \begin{bmatrix} F_0 x_0 \\ F_1 x_1 \\ \vdots \\ F_T x_T \end{bmatrix}$$

where the matrices F_i have some pre-specified elements zero. In particular, if we write

$$F_i x_i = \begin{bmatrix} F_i^1 x_i \\ F_i^2 x_i \\ \vdots \\ F_i^n x_i \end{bmatrix}$$

where F_i^j is the control law applied by the j-th agent at time step i, then those elements of F_i^j are zero that correspond to the elements in the state vector x_i that the j-th agent does not have access to. We can pull the constraints into the state vector and write

$$F_i^j x_i = K_i^j y_i^j,$$

where K_i^j is now a matrix free of any constraints on its elements while the vector y_i^j is a stacked vector of the states of the agents that the j-th agent has access to. This allows us to write

$$F_i x_i = \begin{bmatrix} K_i^1 y_i^1 \\ K_i^2 y_i^2 \\ \vdots \\ K_i^n y_i^n \end{bmatrix}.$$

Thus Λ can be written as

$$\Lambda = \begin{bmatrix} F_0 x_0 \\ F_1 x_1 \\ \vdots \\ F_T x_T \end{bmatrix} = \begin{bmatrix} K_0^1 y_0^1 \\ K_0^2 y_0^2 \\ \vdots \\ K_0^n y_0^n \\ K_1^1 y_1^1 \\ \vdots \\ K_1^n y_1^n \\ \vdots \\ K_T^0 y_T^0 \\ \vdots \\ K_T^n y_T^n \end{bmatrix}. \tag{8}$$

The nT matrices K_i^j are arbitrary and are the optimization variables. Now from (7), we see that the cost function can be written as

$$\begin{aligned} J_T &= E\left[\Gamma'\mathbf{F}\Gamma + \Gamma'\mathbf{G}\Lambda + \Lambda'\mathbf{G}'\Gamma + \Lambda'\mathbf{H}\Lambda\right] \\ &= E\left[\left(\Lambda + H^{-1}G'\Gamma\right)' H \left(\Lambda + H^{-1}G'\Gamma\right)\right] + E\left[\Gamma'\left(GH^{-1}G' + F\right)\Gamma\right]. \end{aligned}$$

The choice of Λ only affects the first term. Thus the optimization problem is

$$\min_{\Lambda} E\left[\left(\Lambda + H^{-1}G'\Gamma\right)' H \left(\Lambda + H^{-1}G'\Gamma\right)\right],$$

where Λ is of the form (8). But this optimization problem is exactly in the form of Proposition 3. Thus we can optimize the value of the cost function. This solves the optimal control law problem.

Remarks:

1) The solution involves the calculation of second order statistic terms which can be calculated off-line since the topology of the network is assumed to be known.
2) The procedure holds even for the case when the topology is time-varying, as long as all the agents know the topology.
3) However note that we need to solve a total of nT coupled matrix equations. This is a

formidable computational burden. In the next subsection, we present a method that is computationally more tractable at the expense of being sub-optimal.

4-3. A Sub-optimal Control Law Algorithm

Once again we note from (5) that the T-horizon cost function to be minimized is

$$J_T = E\left[\sum_{k=0}^{T} u_k' R u_k + \sum_{k=0}^{T} x_k' Q x_k\right] + E\left[x_{T+1}' P_{T+1}^c x_{T+1}\right].$$

We need to choose u_0, u_1, $\cdots$, u_T that minimize J_T. Following (Hassibi *et al.* 1999), we gather terms that depend on the choice of u_K and x_K and write them as

$$\begin{aligned} \Upsilon_T &= E\left[u_T' R u_T + x_T' Q x_T\right] + E\left[x_{T+1}' P_{T+1}^c x_{T+1}\right] \\ &= E\left[\left[\begin{array}{cc} u_T' & x_T' \end{array}\right] \Delta \left[\begin{array}{c} u_T \\ x_T \end{array}\right]\right] + E\left[w_T' P_{T+1}^c w_T\right] \\ &= S_T + O_T \end{aligned}$$

where

$$\begin{aligned} \Delta &= \left[\begin{array}{cc} R + B' P_{T+1}^c B & B' P_{T+1}^c A \\ A' P_{T+1}^c B & Q + A' P_{T+1}^c A \end{array}\right] \\ S_T &= E\left[\left[\begin{array}{cc} u_T' & x_T' \end{array}\right] \Delta \left[\begin{array}{c} u_T \\ x_T \end{array}\right]\right] \\ O_T &= E\left[w_T' P_{T+1}^c w_T\right]. \end{aligned}$$

In the above equation, we have used the system dynamics given in (4) and the fact that the plant noise is zero mean. Thus we can write

$$J_T = E\left[\sum_{k=0}^{T-1} u_k' R u_k + \sum_{k=0}^{T-1} x_k' Q x_k\right] + S_T + O_T. \tag{9}$$

We aim to choose u_T to minimize J_T. From (9), it is clear that the only term where the choice of u_T can make a difference is S_T. On completing squares, S_T can be written as

$$S_T = E\left[\left(u_T - \bar{u}_T\right)' R_{e,T}^c \left(u_T - \bar{u}_T\right)\right] + E\left[x_T' P_T^c x_T\right]$$

where

$$\begin{aligned} R_{e,T}^{c} &= R + B'P_{T+1}^{c}B \\ P_{T}^{c} &= Q + A'P_{T+1}^{c}A - A'P_{T+1}^{c}B\left(R + B'P_{T+1}^{c}B\right)^{-1}B'P_{T+1}^{c}A \end{aligned}$$

and $\bar{u}_T$ is the standard optimal LQ control given by

$$\bar{u}_T = -\left(R_{e,T}^{c}\right)^{-1}B'P_{T+1}^{c}Ax_T.$$

If the controller had access to the entire state, it could simply use the standard optimal control $\bar{u}_T$. However, that is not possible now. Instead, the controller needs to calculate u_T using the information flow that satisfies the topological constraints and choose it to minimize S_T. In other words, we need to find $u_T = F_T x_T$ that minimizes Υ_T where F_T has certain elements zero. The control problem thus reduces to an optimal estimation problem. Once again, we note that

$$u_T = \begin{bmatrix} u_T^1 \\ u_T^2 \\ \vdots \\ u_T^n \end{bmatrix},$$

where each u_T^i is the control law the i-th agent applies and it is a linear function of the measurements the i-th agent has access to. Thus we can write

$$u_T^i = F_T^i x_T,$$

where F_T^i has those elements 0 that correspond to the elements in the state vector x_T that the i-th agent does not have access to. Pulling the constraints into the state vector, we can write

$$u_T^i = K_T^i y_T^i,$$

where K_T^i does not have any constraint while the vector y_T^i is a stacked vector of the states of the agents that the i-th agent has access to. Thus the problem of choosing the control law u_T

reduces to the problem of choosing K_T^i's so as to minimize the criterion

$$E\left[\begin{bmatrix} K_T^1 y_T^1 - \bar{u}_T^1 \\ K_T^2 y_T^2 - \bar{u}_T^2 \\ \vdots \\ K_T^n y_T^n - \bar{u}_T^n \end{bmatrix}' R_{e,T}^c \begin{bmatrix} K_T^1 y_T^1 - \bar{u}_T^1 \\ K_T^2 y_T^2 - \bar{u}_T^2 \\ \vdots \\ K_T^n y_T^n - \bar{u}_T^n \end{bmatrix}\right].$$

This is exactly the optimization problem discussed in Proposition 3. Thus the matrices K_T^i can be easily obtained. Note that this involves solving only n coupled matrix equations and is hence much less computationally expensive than the optimal control law calculation discussed in section 4-2.

Denote the estimation error incurred due to the minimizing choice of u_T by Λ_T. We have

$$S_T = \Lambda_T + E\left[x_T' P_T^c x_T\right].$$

We can thus write the cost function as

$$\begin{aligned} J_T &= E\left[\sum_{k=0}^{T-1} u_k' R u_k + \sum_{k=0}^{T-1} x_k' Q x_k\right] + S_T + O_T \\ &= E\left[\sum_{k=0}^{T-1} u_k' R u_k + \sum_{k=0}^{T-1} x_k' Q x_k\right] + \Lambda_T + E\left[x_T' P_T^c x_T\right] + O_T \\ &= J_{T-1} + \Lambda_T + O_T. \end{aligned}$$

Thus we now need to choose control inputs for time steps 0 to $T-1$ to minimize J_T. By scanning the terms on the right hand side of the equation, we see that O_T is independent of the choice of control laws from time 0 to $T-1$. However, unlike the standard case of control with imperfect observations (Hassibi *et al.* 1999), we note that apart from J_{T-1}, the estimation error Λ_T is also a function of the state x_T and hence of the (unknown) control law u_{T-1}. Moreover it is a non-linear function of u_{T-1}. Thus the control u_{T-1} should be chosen to minimize the cost $J_{T-1} + \Lambda_T$. Thus, the separation principle does not hold in general. This is related to the fact that the information pattern is not classical (see, e.g., (Witsenhausen 1971)) because the previous control law is not known fully to all the agents. We get across this problem by neglecting the estimation cost Λ_T and optimizing only J_{T-1}. For this purpose, we note that our argument so far was independent of time index T. Thus we can recursively apply the argument for time steps $T-1$, $T-2$ and so on.

Remarks:

1) We have enforced a separation principle artificially that says that the controller synthesis problem can be separated into an estimation problem and the usual LQR control problem. At every time step, every controller tries to estimate the optimal control law from the information it has access to (in the sense of Proposition 3) and uses this estimate in the optimal LQR control law.
2) This method is in general sub-optimal since the separation principle does not hold in reality. However since this method replaces solution of nT coupled matrix equations by solving n coupled matrix equations T times, this method saves a lot on computational cost.
3) If needed, better performance can be achieved by including the estimation cost Λ_T in calculation of u_{T-1}. It can be proved that this inclusion results in a convex problem that can be solved efficiently. However this method would still not be optimal since for calculation of u_{T-2}, we need to consider J_{T-2}, Λ_{T-1} and the cost incurred in imperfectly minimizing Λ_T. Thus the problem starts involving more and more terms to optimize over. The extent of sub-optimality can be reduced by including more terms in the optimization.
4) Intuitively, the approximation can be thought of as follows. At any time, the optimal control input of an agent will depend on the control inputs of all other agents at the previous time step. However the agent is not allowed to observe these. We get around this problem by ignoring the direct dependence of the optimal control input on these terms. Instead, we use the fact that these terms will soon show up in the values of the states of the neighbors of the agent, which are being observed. Thus these terms will eventually be used in the calculation of control inputs.
5) For the fully connected topology, the sub-optimal algorithm yields the same result as the optimal algorithm since there is no estimation cost Λ_T. In other words, since the control inputs of all the agents can be calculated whenever needed, there is no approximation involved in ignoring their effect on each agent's control input.

5. EXAMPLES

We now consider two examples to illustrate the issues involved.

Example 1: Consider a network of four agents, each with single integrator dynamics. This case is of interest since single integrator dynamics can be used to solve consensus problems. Let the agents be designated as v_i, $i = 1, 2, 3, 4$. The agent v_i has dynamics

$$\begin{aligned} x^i_{k+1} &= x^i_k - 0.2u^i_k + w^i_k \\ u^i_k &= F^{i,1} x^i_k + \sum_{\text{all out-neighbors } j} F^{ij,2}(x^j_k - x^i_k). \end{aligned}$$

We denote x_k to be the state of the whole system, where

$$x_k = \left[x^1_k, x^2_k, x^3_k, x^4_k\right]'.$$

Similarly denote u_k to be the control vector obtained by stacking all the u^i_k's. Then the evolution of the system is described as

$$\begin{aligned} x_{k+1} &= x_k - 0.2u_k + w_k \\ u_k &= F^1 x_k + F^2 x_k, \end{aligned}$$

where F^1 is a diagonal matrix with $F^{1,1}, F^{2,1}, F^{3,1}, F^{4,1}$ as the diagonal elements; and the $(i, j)-$th element of the matrix F^2 is given by

$$[F^2]_{i,j} = \begin{cases} F^{ij,2} & i \neq j \text{ and } j \text{ is an out-neighbor of } i \\ 0, & i \neq j \text{ and } j \text{ is not an out-neighbor of } i \\ -\sum_j F^{ij,2}, & i = j. \end{cases}$$

The initial condition is random with zero mean and covariance as identity matrix. Similarly the noise is white Gaussian with zero mean and covariance as identity matrix. The cost function specified is

$$J = \sum_{k=0}^{T} E\left[x'_k Q x_k + u'_k R u_k\right].$$

We present results for $T = 30$. We take the weighting matrices arbitrarily to be as follows:

$$Q = \begin{bmatrix} 1.6158 & 1.6884 & 1.2138 & 0.563 \\ 1.6884 & 2.798 & 1.2843 & 1.2528 \\ 1.2138 & 1.2843 & 0.9645 & 0.5147 \\ 0.563 & 1.2528 & 0.5147 & 0.7501 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

First we note that if all the agents are communicating with one another, the sub-optimal and optimal algorithm give the same cost and the control law matrix. We also consider a constrained topology where we allow limited communication to happen. The topology is as follows. The vehicle v_1 can talk to v_2, the vehicle v_2 to v_1 and v_3, the vehicle v_3 to v_2 and v_4 and v_4 can talk to v_3. In this case, the evolution of the cost is shown in figure 1. We can see that the loss in performance from the sub-optimal algorithm is not huge. The savings in computational time are considerable, however. Note that at the intermediate time values, the sub-optimal algorithm is performing better than the optimal algorithm. However, this can be easily explained by noting that the optimal algorithm is optimal for a time horizon of 30 steps and there is no guarantee that it is the optimal algorithm for a smaller time window as well.

In figure 2 we show the steady state cost for the ring topology for a time horizon of 100 time steps for the ring topology as we introduce delay into the system. The ring topology involves all communication links being present, except the v_2v_4 and v_1v_3 links. We assume that the state information is passed with some delay as a multiple of sampling time of the system but the agents calculate the control law assuming there is no delay. It can be seen that the cost slowly increases and the system is reasonably robust to delay uncertainty. It becomes unstable only for a delay equal to or greater than 5 time steps.

Example 2: In this example we use the dynamics of each agent as the dynamics of the Caltech Multi Vehicle Wireless Testbed vehicles, as described in (Cremean *et al.* 2002, Waydo

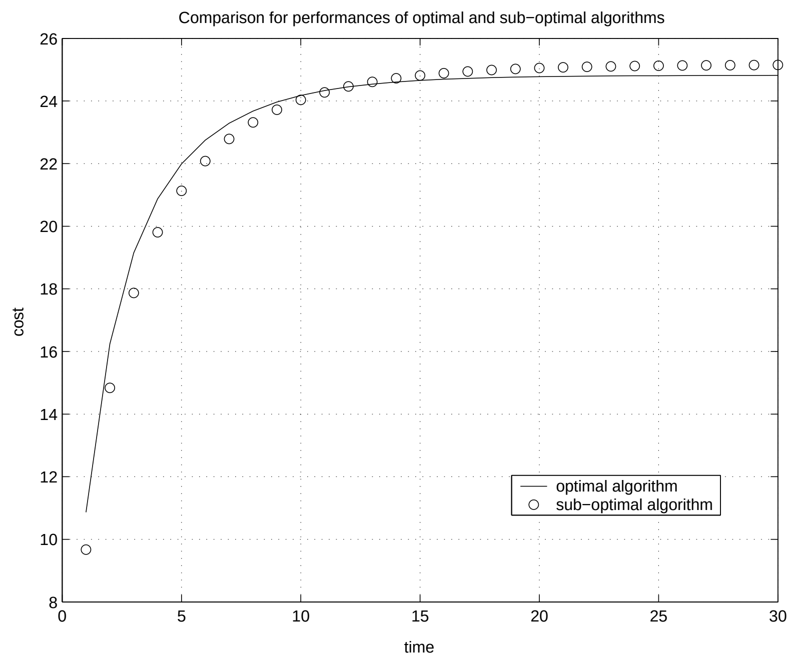

Fig. 1. The loss in performance due to the sub-optimal algorithm is not huge. Cost considered is $E\left[x_k' Q x_k + u_k' R u_k\right]$.

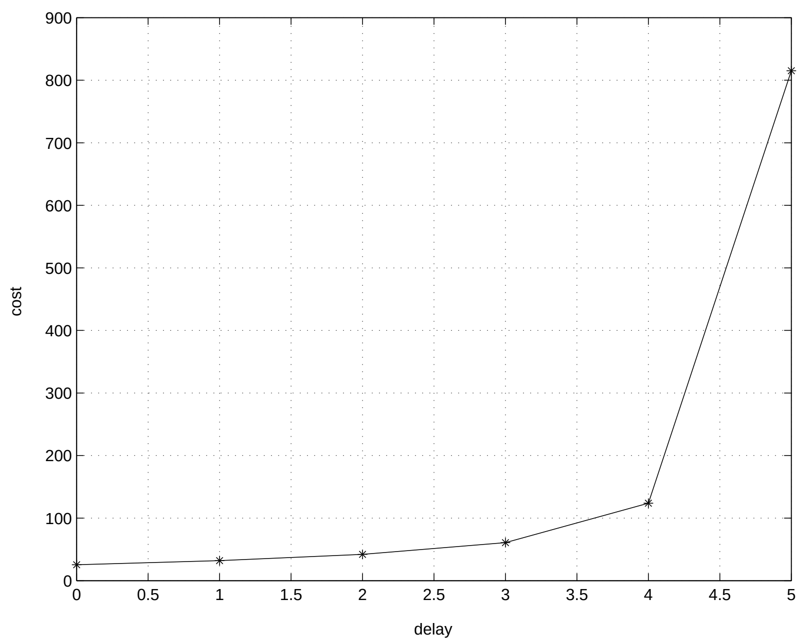

Fig. 2. The sub-optimal algorithm is robust to delays. Cost considered is $E\left[x_k' Q x_k + u_k' R u_k\right]$.

et al. 2004). The non-linear dynamics are given by

$$\begin{aligned}
m\ddot{x} &= -\mu\dot{x} + (F_L + F_R)\cos(\theta) \\
m\ddot{y} &= -\mu\dot{y} + (F_L + F_R)\sin(\theta) \\
J\ddot{\theta} &= -\psi\dot{\theta} + (F_R - F_L)r_f.
\end{aligned}$$

F_L and F_R are the inputs, $m = 0.749$kg is the mass of vehicle, $J = 0.0031$kg m^2 is the moment of inertia, $\mu = 0.15$ kg-s is the linear frictional coefficient, $\psi = 0.005$kgm^2/s is the rotational friction coefficient and $r_f = 0.089$m is the distance from the center of mass of the vehicle to the axis of the fan. On linearizing the dynamics about the straight line $y = x$ at a velocity of 1ms^{-1} along the x and y axes, we obtain the equations

$$\begin{aligned}
\dot{X} &= AX + BU \\
U &= FX,
\end{aligned}$$

where

$$X = \begin{bmatrix} x & y & \theta & \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}'$$

$$A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{-(F_L^{nom}+F_R^{nom})\sin(\theta^{nom})}{m} & \frac{-\mu}{m} & 0 & 0 \\
0 & 0 & \frac{(F_L^{nom}+F_R^{nom})\cos(\theta^{nom})}{m} & 0 & \frac{\mu}{m} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-\psi}{J}
\end{bmatrix}$$

$$B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\frac{\cos(\theta^{nom})}{m} & \frac{\cos(\theta^{nom})}{m} \\
\frac{\sin(\theta^{nom})}{m} & \frac{\sin(\theta^{nom})}{m} \\
\frac{-r_f}{J} & \frac{-r_f}{J}
\end{bmatrix}$$

$$\theta^{nom} = \frac{\pi}{4} \qquad\qquad F_L^{nom} = F_R^{nom} = \frac{\mu}{\sqrt{2}}.$$

We discretize the above equations with a step size $h = 0.2$. We consider 8 vehicles starting from an octagonal formation and consider the topologies possible as the communication radius of each vehicle is increased. It is apparent that by symmetry there are 5 distinct topologies possible, with each vehicle talking to 0, 2, 4, 6 and 7 other vehicles respectively. The initial covariance matrix R_0 is the identity matrix. The cost function matrix R is also identity while the matrix Q is randomly generated. The cost function horizon is $T = 100$ time steps. A typical curve for the varying of the costs provided by the sub-optimal algorithm as the communication radius is increased is given in figure 3. Following general conclusions can be drawn for the

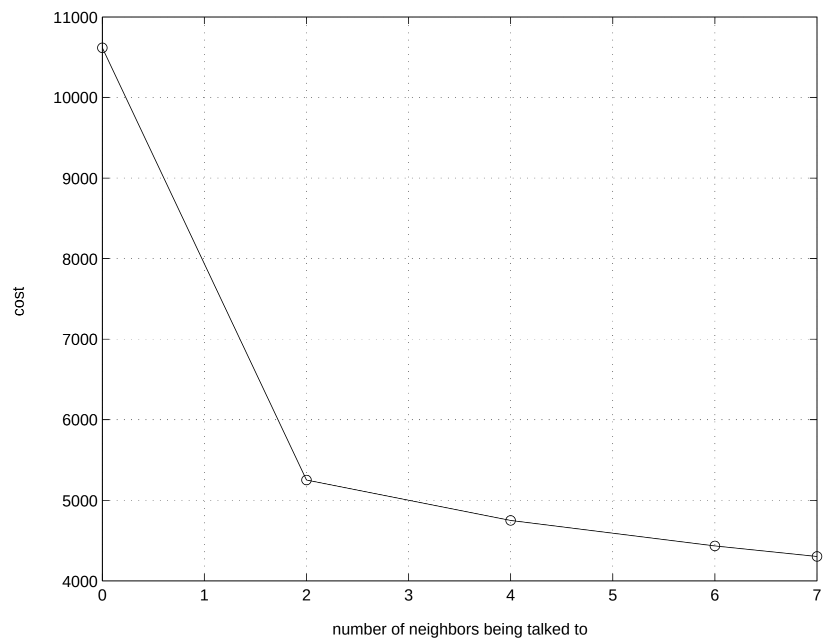

Fig. 3. As the communication radius is increased, the cost goes down. Cost considered is $E\left[x_k' Q x_k + u_k' R u_k\right]$.

example from the plot.

1) As more and more communication is allowed, the cost goes down.
2) The marginal utility of each communication link decreases as more and more links are added.

The difference in the performance between the sub-optimal and the optimal algorithms increased as the communication topology became more and more sparse. Figure 4 shows another plot comparing the comparison of optimal and sub-optimal algorithms for a different value of the Q matrix. It can be seen that even for the decentralized case, the error is of the order of only 30%.

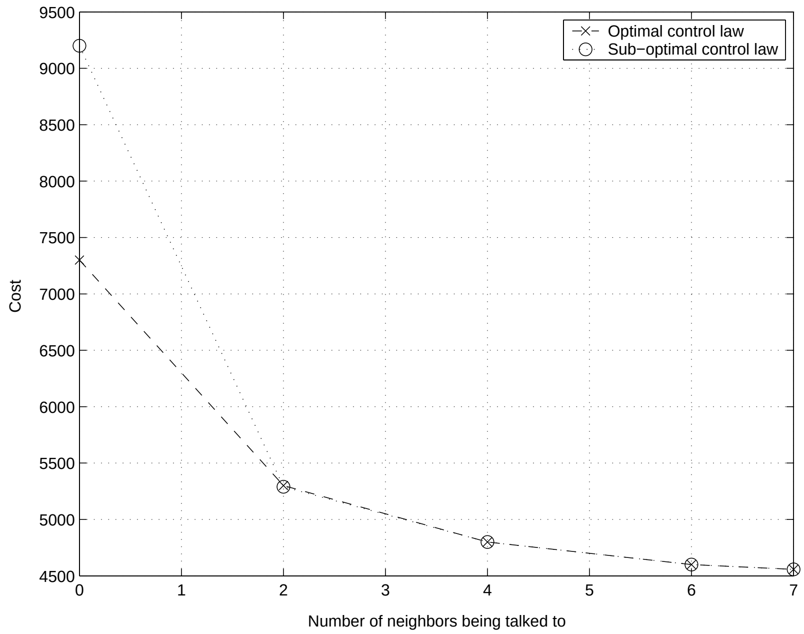

Fig. 4. As the communication radius is increased, loss in performance due to the sub-optimal algorithm decreases. Cost considered is $E\left[x_k' Q x_k + u_k' R u_k\right]$.

6. Conclusions and Future Work

In this paper, motivated by synthesis of optimal control laws for interconnected network of agents, we considered the problem of synthesis of a LQR optimal control law which is constrained to lie in a particular vector space. We saw that the problem was hard to solve in general. We presented a computationally expensive method for the optimal finite time horizon control and a computationally easier method to generate a sub-optimal control law. We presented examples which illustrated that the loss in performance due to the sub-optimal algorithm is not huge and that communication in general helps to bring down the cost. The methods involve the solution of linear equations and are thus free from convergence problems.

Although this work is a significant advance in the field in many respects, more work is needed to fully understand and solve the problem of optimal control of a network of dynamic agents. The optimal control law we have presented serves as a good benchmark to evaluate any other control strategy; however, it is computationally very expensive and good approximations that are more tractable will be useful in many situations. We have presented one such sub-optimal algorithm. From numerical examples it seems that the loss in performance is not huge. However, we have not been able to obtain an analytic expression of the loss in performance or a bound on it. As discussed in the paper, for a fully connected topology, there is no loss of performance. Moreover plots like Figure 4 suggest that as the topology becomes more and more sparse, the loss in performance increases. Work characterizing the performance of the various synthesis algorithms as a function of topology is also likely to aid us in *designing* topologies that yield better performance. In general, the optimal cost is achieved by the fully centralized topology. We might, however, be interested in putting additional constraints on the topology, e.g., the total number of links might be limited to prevent congestion in the shared channel or network. Thus a more relevant question is to obtain *a* topology for which there exists a control law such that the cost achieved is at most a prescribed constant α times the cost achieved by the fully centralized topology. An analytic method for obtaining a relation between topology and optimal cost is needed. Finally work characterizing the effect of topology on the cost will also help in understanding the robustness of the algorithms to knowledge of the topology. Currently, we assume that either the control law is calculated off-line by a central processor, or each node knows the topology and calculates the control law for the entire system and extracts its own control law from it. To make this implementation more scalable, it will be useful to understand the effects of topology changes far from the individual agent, or equivalently, to imperfect knowledge of the topology far away.

There are more areas that this work can be extended to model more real-life situations. One is to remove the assumption of perfect communication when a link exists. Although our algorithms represent an improvement over many methods existing in the literature since they can handle a time-varying topology, we still assume that the topology is known at every time step. This would be the case if the change is deterministic. Usual communication models are stochastic in nature, e.g., the loss of a link can be modeled as a random process. Extending our algorithms to such stochastic models (e.g., based on the work (Nilsson 1998)) is an important direction we are currently pursuing.

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REFERENCES

Anderson, B. D. O. and J. B. Moore (1971). *Linear Optimal Control*. Prentice-Hall.

Antsaklis, P. and J. Baillieul (2004). Editors, special issue on networked control systems. *IEEE Transactions on Automatic Control*.

Bamieh, B., F. Paganini and M. A. Dahleh (2002). Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control* **47**(7), 1091–1107.

Bemporad, A., M. Morari, V. Dua and E. N. Pistikipoulos (2002). The explicit linear quadratic regulator for constrained systems. *Automatica* **38**(1), 3–20.

Biggs, N. (1974). *Algebraic Graph Theory*. Cambridge Tracks in Mathematics. Cambridge University Press.

Blondel, V., M. Gevers and A. Lindquist (1995). Survey on the state of systems and control. *European Journal of Control* **1**, 5–23.

Cremean, L., B. W. Dunbar, D. van Gogh, J. Hickey, E. Klavins, J. Meltzer and R. M. Murray (2002). The Caltech multi-vehicle wireless testbed. In: *Proc of the 2002 Conference on Decision and Control*.

D'Andrea, R. and G. E. Dullerud (2003.). Distributed control design for spatially interconnected systems. *IEEE Transactions on Automatic Control* **48**(9), 1478–1495.

Darbha, S., S. Pargaonkar and S. P. Bhattacharya (2004). A linear programming approach to the synthesis of fixed structure controllers. In: *Proceedings of the 2004 American Control Conference*.

de Castro, G. A. (2003). Convex Methods for the design of Structured Controllers. PhD thesis. University of California, Los Angeles.

Dunbar, W. B. and R. M. Murray (2004). Distributed receding horizon control with applications to multi-vehicle formation stabilization. *Automatica*. Submitted.

Fan, C., J. L. Speyer and C. R. Jaensch (1994). Centralized and decentralized solutions of the linear-exponential-gaussian problem. *IEEE Transactions on Automatic Control* **39**(10), 1986–2003.

Fax, J. A. (2001). Optimal and Cooperative Control of Vehicle Formations. PhD thesis. California Institute of Technology.

Fax, J. A. and R. M. Murray (2004). Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control* **49**(9), 1465–1476.

Franco, E., T. Parisini and M. Polycarpou (2004). Cooperative control of discrete-time agents with delayed information exchange: a receding horizon approach. In: *Proceedings of the 2004 IEEE Control and Decision Conference*.

Grieder, P., F. Borrelli, F. Torrisi and M. Morari (2003). Computation of the constrained infinite time linear quadratic regulator. In: *Proc. of the American Control Conference*.

Gupta, V., B. Hassibi and R. M. Murray (2003). Stability analysis of stochastically varying formations of dynamic agents.. In: *Proc. of IEEE Conf. on Decision and Control*.

Gupta, V., B. Hassibi and R. M. Murray (2004). On the synthesis of control laws for a network of autonomous agents. In: *Proc. of the American Control Conference*.

Hassibi, B., A. H. Sayed and T. Kailath (1999). *Indefinite-Quadratic Estimation and Control*. Studies in Applied and Numerical Mathematics.

Jadbabaie, A., J. Lin and A. S. Morse (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control* **48**(6), 988–1001.

Kailath, T., A. H. Sayed and B. Hassibi (2000). *Linear Estimation*. Prentice Hall.

Kleinman, D. L., T. Fortmann and M. Athans (1968). On the design of linear systems with piecewise-constant feedback gains. *IEEE Transactions on Automatic Control* **AC-13**(4), 354–361.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems*. John Wiley and Sons.

Lancaster, P. (1969). *Theory of Matrices*. Academic Press.

Langbort, C., R. S. Chandra and R. D'Andrea (2004). Distributed control design for systems interconnected over an arbitrary graph. *IEEE Transactions on Automatic Control* **49**(9), 1502–1519.

Levine, W. S., T. L. Johnson and M. Athans (1971). Optimal limited state variable feedback controllers for linear systems. *IEEE Transactions on Automatic Control* **AC-16**, 785–793.

Nilsson, J. (1998). Real-Time Control Systems with Delays. PhD thesis. Department of Automatic Control, Lund Institute of Technology.

Papadimitriou, C. H. and J. N. Tsitsiklis (1986). Intractable problems in control theory. *SIAM Journal on Control and Optimization* **24**, 639–654.

Ren, W. and R. W. Beard (2004). Consensus of information under dynamically changing interaction topologies. In: *Proc. of the American Control Conference*. pp. 4939–4944.

Rotkowitz, M. and S. Lall (2002). Decentralized control information structures preserved under feedback. In: *Proc. of IEEE Conf. on Decision and Control*.

Saber, R. O. and R. M. Murray (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control* **49**, 1520–1533.

Soderstrom, T. (1978). On some algorithms for design of optimal constrained regulators. *IEEE Transactions on Automatic Control* **AC-23**, 1100–1101.

Syrmos, V. L., C. T. Abdullah, P. Dorato and K. Grigoriadis (1997). Static output feedback - a survey. *Automatica* **33**(2), 125–137.

Waydo, S., Z. Jin, E. Wildanger, M. Lammers, H. Scholze, P. Foley, R. M. Murray and D. Held (2004). MVWT-II: The second generation caltech multi-vehicle wireless testbed. In: *American Control Conference*.

Wenk, C. J. and C. H. Knapp (1980). Parameter optimization in linear systema with arbitrarily constrained controller structure. *IEEE Transactions on Automatic Control* **AC-25**(3), 496–500.

Witsenhausen, H. S. (1968). A counterexample in stochastic optimum control. *SIAM Journal of Control* **6**(1), 131–147.

Witsenhausen, H. S. (1971). Separation of estimation and control for discrete time systems. *Proceedings of the IEEE* **59**(11), 1557–1566.

Xiao, L. and S. Boyd (2004). Fast linear iterations for distributed averaging. *System and Control Letters* **53**, 65–78.

Figure Captions

- Fig 1. The loss in performance due to the sub-optimal algorithm is not huge. Cost considered is $E\left[x_k' Q x_k + u_k' R u_k\right]$.
- Fig 2. The sub-optimal algorithm is robust to delays. Cost considered is $E\left[x_k' Q x_k + u_k' R u_k\right]$.
- Fig 3. As the communication radius is increased, the cost goes down. Cost considered is $E\left[x_k' Q x_k + u_k' R u_k\right]$.
- Fig 4. As the communication radius is increased, loss in performance due to the sub-optimal algorithm decreases. Cost considered is $E\left[x_k' Q x_k + u_k' R u_k\right]$.