Equalization with an H^∞ Criterion *

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Abstract

In this paper we study the problem of equalizing a communications channel from an H^∞ point of view. H^∞ estimation theory has been recently introduced as a method for designing filters that have acceptable performance in the face of model uncertainty and lack of statistical information on the exogenous signals. The approach may therefore be attractive for channel equalization problems where rapid changes in the system parameters lead to such model uncertainty, and where conventional statistical methods may not be directly applicable.

In this paper we obtain a closed-form solution to the H^∞ equalization problem and parametrize all possible H^∞-optimal equalizers. In particular, we show that, for minimum phase channels, a scaled version of the zero-forcing equalizer is H^∞-optimal. The results also indicate an interesting dichotomy between minimum phase and non-minimum phase channels: for minimum phase channels the best causal equalizer performs as well as the best noncausal equalizer, whereas for non-minimum phase channels, causal equalizers cannot reduce the estimation error bounds from their a priori values. [In other words, causal equalization of non-minimum phase channels is not possible.] Our analysis also suggests certain remedies in the non-minimum phase case, namely, allowing for finite delay, oversampling, or using multiple sensors. For example, we show that H^∞ equalization of non-minimum phase channels requires a time delay of at least d units, d is the number of non-minimum phase zeros of the channel.

Index Terms: Equalization, H^∞ estimation, robustness, minimum phase channels.

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1 Introduction

Equalization is a well studied problem in the area of communications. Various structures and methods have been proposed to recover transmitted data from their filtered and noise corrupted versions, and each method has its own advantages and disadvantages in terms of performance and complexity [1]. Equalization is also closely related to estimation theory since the equalization problem can be considered as a special case of an estimation problem.

Recently, in estimation theory, a (so-called) H^∞ approach has been proposed and extensively studied (see *e.g.*, [2, 3, 4, 5, 6, 7] and the references therein) with the belief that the resulting H^∞ estimators will be more robust with respect to model uncertainties and lack of statistical knowledge of the exogenous signals. Due to its aforementioned properties, the H^∞ approach has also been proposed as an alternative method for channel equalization [8]. Further study of this approach to the equalization problem is the main focus of this article. As we shall shortly see, the results obtained in this attempt provide us with a new and different perspective for the understanding and analysis of the equalization problem, as well as for H^∞ estimation itself.

The remainder of this paper is organized as follows. In Sec.2 we introduce the equalization problem, as well as the issue of robustness with respect to model uncertainty and lack of statistical information. Sec. 3 introduces the H^∞ framework for robust estimation and shows how the H^∞ estimation problem can be solved via a ceratin J-spectral factorization. (The proof of this result is provided in App. A). Sec. 5 contains the main results of this paper. The H^∞ equalization problem is solved and all solutions are parametrized. The derivation hinges on the explicit J-spectral factorization of Sec. 5.1.1 and shows that the solution depends on whether the channel is minimum phase or not. It is also shown that, for minimum phase channels, appropriately scaled zero-forcing equalizers are H^∞-optimal. Moreover, the relative merits of various H^∞-optimal equalizers are discussed. Sec. 5.2 deals with the non-square case and Sec. 6 studies some remedies for non-minimum phase channels. In particular, it is shown that successful equalization of non-minimum phase channels requires a time delay greater than or equal to the number of non-minimum phase zeros. Sec. 7 gives state-space formulae for the H^∞ equalizers and the paper concludes with Sec. 8.

2 The Equalization Problem

The equalization problem that will be studied in this paper is depicted in Fig. 2. In this framework $\{u_i\}$ is an unknown sequence (the *transmitted sequence*), $\{v_i\}$ is an unknown additive disturbance sequence,

$\{y_i\}$ is a known *observations* sequence, and $H(z)$ is a known linear time-invariant (LTI) communications channel. The goal is to design the linear time-invariant system $K(z)$ (the so-called *equalizer*) so as to estimate the unknown transmitted sequence $\{u_i\}$ from the known observations sequence $\{y_i\}$, where the estimated sequence is denoted by $\{\hat{u}_i\}$. In this sense, the equalization problem is essentially one of *inverting* a noisy channel (to obtain the input, from noisy observations of the output of the channel).

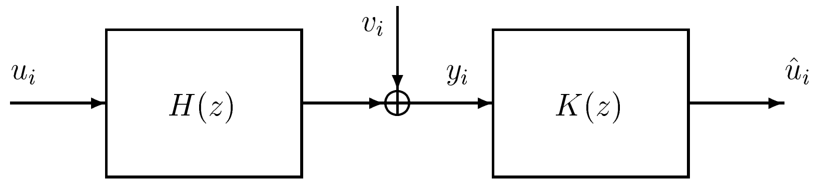

Figure 1: The equalization problem.

In communications applications, the transmitted sequence $\{u_i\}$ is often chosen from a finite alphabet. In this paper, however, we shall not impose such a restriction and shall assume that the u_i are arbitrary vectors in $\mathcal{C}^m$. The unknown additive disturbance $\{v_i\}$ is typically composed of three components: (i) measurement noise, (ii) interference from other signals, and (iii) modeling errors, due to imperfect knowledge of the *true* channel. We shall therefore, for the most part, not make any statistical assumptions on the disturbance sequence $\{v_i\}$ and will simply consider it as an unknown sequence of elements in $\mathcal{C}^p$. The observations $\{y_i\}$ will therefore also be a sequence of elements in $\mathcal{C}^p$.

The dimensions of $\{u_i\}$ and $\{y_i\}$ determine the input-output dimensions of the channel $H(z)$. The channel is assumed to be causal and stable, which means that it is a $p \times m$ matrix function in z with Laurent series expansion,

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots \tag{1}$$

that is analytic on and outside the unit circle, $|z| = 1$, where $\{h_i, i \geq 0\}$ denotes the impulse response of $H(z)$. The channel $H(z)$ is often assumed to be rational (of McMillan degree, say, n), or in fact finite-impulse-response (FIR, of length, say, n). However, almost all our results extend to the nonrational case, and we shall therefore not make this restriction.

The equalizer $K(z)$ is assumed to be linear time-invariant since the channel is so, and since $u_i \in \mathcal{C}^m$. [When the transmitted sequence is from a finite alphabet then having a *nonlinear* equalizer is justified, say by appending a decision device to $K(z)$, or using a decision-feedback or maximum-likelihood equalizer.] Although we shall assume that $K(z)$ is stable, we shall not always assume that it is causal. Indeed for the purposes of this paper, it will be important to distinguish between the following two cases.

- **Causal Equalizer:** The estimate $\hat{u}_i$ uses only past and current observations, $\{\ldots, y_{i-2}, y_{i-1}, y_i\}$. Thus, we may write

$$K(z) = k_0 + k_1 z^{-1} + k_2 z^{-2} + \ldots \tag{2}$$

where $\{k_i, i \geq 0\}$ denotes the corresponding impulse response. Therefore $K(z)$ is a $m \times p$ matrix function that is analytic on and outside the unit circle, $|z| = 1$. In the rational case, this means that $K(z)$ has all its poles inside the unit circle.

- **Noncausal Equalizer:** The estimate $\hat{u}_i$ uses past, current, *and future* observations, $\{\ldots, y_{i-1}, y_i, y_{i+1}, \ldots\}$. Thus, we may write

$$K(z) = \ldots + k_{-1} z + k_0 + k_1 z^{-1} + \ldots \tag{3}$$

where $\{k_i, -\infty \leq i \leq \infty\}$ denotes the corresponding impulse response. Therefore $K(z)$ is a $m \times p$ matrix function that is analytic on some annulus containing the unit circle, $|z| = 1$. In the rational case, this means that $K(z)$ has poles *both* inside and outside the unit circle.

We shall also briefly consider the case where $\hat{u}_i$ uses a finite number of future observations, say $\{\ldots, y_{i+d-1}, y_{i+d}\}$, for some $d > 0$, in Sec. 6.3.

Finally, we should mention that we shall mostly be concerned with the case where $H(z)$ is square, *i.e.*, $m = p$, and often with the case where it is scalar, *i.e.*, $m = p = 1$.

2.1 The Question of Robustness

The equalization problem is, of course, a central problem in communications and therefore a variety of approaches and methods exist for solving it. We may mention minimum-mean-square-error (MMSE), zero-forcing (ZF), and, in passing, maximum-likelihood-sequence-detection (MLSD, a.k.a. as the Viterbi algorithm) and decision-feedback (DF) equalizers [9, 10, 11, 12, 13, 1].

MMSE equalizers assume that the unknown sequences $\{u_i\}$ and $\{v_i\}$ are zero-mean stationary processes with known second-order statistics, and choose $K(z)$ so as to minimize the expected squared estimation error, $E(u_i - \hat{u}_i)^*(u_i - \hat{u}_i)$. When $\{u_i\}$ and $\{v_i\}$ are uncorrelated and white zero-mean stationary process with variances Q and R, respectively, then the noncausal MMSE equalizer is given by the so-called Wiener smoother,

$$K(z) = QH^*(z^{-*}) \left[R + H(z)QH^*(z^{-*})\right]^{-1}, \tag{4}$$

and the causal MMSE equalizer by the Wiener filter, which in this case can be shown to be,

$$K(z) = Qh_0^* R_e^{-1} M^{-1}(z), \tag{5}$$

where the monic ($M(\infty) = I_p$) and minimum phase[1] matrix function $M(z)$, and the Hermitian matrix $R_e = R_e^* \in \mathcal{C}^{p\times p}$, are found from the spectral factorization,

$$M(z)R_e M^*(z^{-*}) = R + H(z)QH^*(z^{-*}). \tag{6}$$

In zero-forcing equalization, assuming $p = m$, $K(z)$ is simply chosen as the inverse of the channel, *i.e.*,

$$K(z) = H^{-1}(z). \tag{7}$$

Clearly, the zero-forcing equalizer is causal if $H(z)$ is minimum phase and noncausal if it is otherwise. When there is no additive disturbance present ($\{v_i\} = \{0\}$), the zero-forcing equalizer perfectly reconstructs the transmitted sequence $\{u_i\}$. However, when $\{v_i\}$ is present it has the drawback of significantly enhancing the noise at those frequencies for which $H(z)$ has deep nulls.

The Viterbi algorithm assumes that the $\{u_i\}$ are binary (or M-ary) sequences with uniform distribution and that $\{v_i\}$ is white Gaussian noise, and obtains maximum-likelihood estimates of the transmitted sequence. Both the Viterbi algorithm and decision-feedback equalizers are nonlinear and will not be further considered in this paper. (We should remark that nonlinear H^∞ estimators are also possible, though we shall not consider them in this paper.)

With such an abundance of equalizers it is natural to ask what more is there to say? In fact, there is quite more if we note that MMSE and MLSD equalizers have certain desirable optimality properties, as suggested by their names, only when we assume:

(i) that the channel $H(z)$ is known exactly.

(ii) that the first and second order statistics (for MMSE) and the complete distributions (for MLSD) of the underlying disturbances are known exactly.

In practice, both the above assumptions are often violated. $H(z)$ is typically not known beforehand and must be determined via some identification procedure (say, by transmitting training sequences). This invariably leads to modeling errors, which must be relegated to the additive dusturbance $\{v_i\}$. Once $\{v_i\}$ has components due to modeling errors and interference, it is difficult to determine what statistics, or distributions, it may have (or whether it is truly stochastic at all).

[1] By a minimum phase matrix function $M(z)$, we mean one which is causal and causally invertible, *i.e.*, one for which $M(z)$ and $M^{-1}(z)$ are both analytic on $|z| \geq 1$.

Therefore a natural question to ask is what the performance of such equalizers will be if the assumptions on the disturbances are violated, or if we have modeling errors? In other words:

- *is it possible that* **small** *disturbances and modeling errors may lead to* **large** *estimation errors?*

Intuitively, a non-robust equalizer is one for which the above is true, *i.e.*, one for which small disturbances may lead to large estimation errors. A robust equalizer is one for which small disturbances lead to small estimation errors.

The problem of robust equalization, or, more fundamentally, robust estimation, is an important one. In the next section, we shall describe the H^∞ estimation framework which is an *attempt* at addressing the robusntess question. Any approach to robust estimation requires a *measure* of largeness and smallness (for the signals involved), and in the H^∞ framework this measure is *energy.* The idea is to come up with estimators that minimize (or in the suboptimal case, bound) the maximum energy gain from the disturbances to the estimation errors. This will guarantee that if the disturbances are small (in energy) then the estimation errors will be as small as possible (in energy), *no matter what the disturbances are.* The robustness of the H^∞ estimators arises from this fact. However, since they make no assumption about the disturbances and have to accommodate for all conceivable ones, they may be over-conservative. (For more on H^∞ estimation see [2, 3, 4, 5, 6, 7] and the references therein.)

3 H^∞ Estimation

A general estimation problem is shown in Fig. 3, where $\{u_i \in \mathcal{C}^m\}$ is an unknown input sequence, $\{v_i \in \mathcal{C}^p\}$ is an unknown additive disturbance sequence, $\{y_i \in \mathcal{C}^p\}$ is a known measurement sequence, and $H(z)$ and $L(z)$ are known causal and stable linear time-invariant systems. The goal is to construct the linear time-invariant $K(z)$ (called the estimator) to estimate the unobservable desired sequence $\{s_i \in \mathcal{C}^q\}$ from the observations $\{y_i\}$. The estimates are denoted by $\{\hat{s}_i\}$. Clearly, the equalization problem is a special case of this formulation with $L(z) = I_m$.

The behavior of any estimator $K(z)$ can be captured by the induced transfer matrix, say $T_K(z)$, that maps the unknown disturbances $\{u_i\}$ and $\{v_i\}$ to the estimation errors $\{s_i - \hat{s}_i\}$:

$$\mathcal{T}_{\mathcal{K}} : \left\{ \begin{bmatrix} u_i \\ v_i \end{bmatrix} \right\} \to \{s_i - \hat{s}_i\}. \tag{8}$$

Now using Fig. 3, we readily see

$$T_K(z) = \begin{bmatrix} L(z) - K(z)H(z) & -K(z) \end{bmatrix}. \tag{9}$$

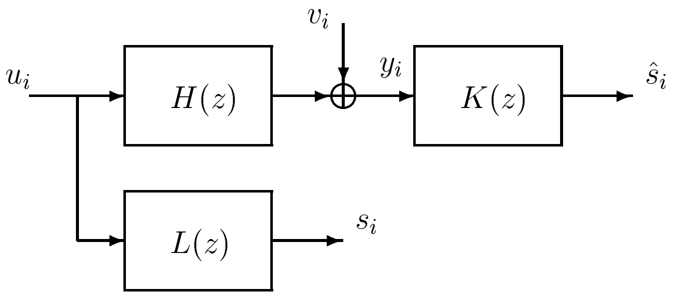

Figure 2: A general estimation problem.

In estimation our goal is to make the transfer matrix $T_K(z)$ small in some sense. For example, in MMSE estimation the (so-called) H^2 norm of $T_K(z)$ is minimized, where

$$\|T_K(z)\|_2 \triangleq \left(\frac{1}{2\pi} \int_0^{2\pi} \text{trace}\left[T_K(e^{j\omega})T_K^*(e^{j\omega})\right] d\omega \right)^{1/2}. \tag{10}$$

In H^∞ estimation the goal is to choose $K(z)$ to minimize the H^∞ norm of $T_K(z)$.

Definition 1 (The H^∞ Norm) *Suppose that the stable linear time-invariant system with transfer matrix $T(z)$ maps the input sequence $u = \{u_i\}$ to the output sequence $y = \{y_i\}$. Then the H^∞ norm of $T(z)$ is defined as*

$$\|T(z)\|_\infty \triangleq \sup_{u \in l^2, u \neq 0} \frac{\|y\|_2}{\|u\|_2} \tag{11}$$

where $\|a\|_2^2 \triangleq \sum_{i=-\infty}^{\infty} a_i^* a_i$ and l^2 denotes the space of all square-summable sequences.

The H^∞ norm has the following wellknown frequency domain characterization,

$$\|T(z)\|_\infty = \sup_{0 \leq \omega \leq 2\pi} \bar{\sigma}\left[T(e^{j\omega})\right] = \sup_{0 \leq \omega \leq 2\pi} \lambda_{\max}^{1/2}\left[T(e^{j\omega})T^*(e^{j\omega})\right], \tag{12}$$

where $\bar{\sigma}(\cdot)$ denotes the maximum singular value of its argument. Thus the H^∞ norm, which is the square-root of the maximum input-output energy gain, is simply the peak of the frequency response, or the maximum amplification obtained when applying sinusoidal signals to $T(z)$.

The H^∞ estimation problem can thus be formulated as follows.

Problem 1 (Optimal H^∞ Estimation Problem) *Find a causal estimator $K(z)$ that satisfies*

$$\inf_{K(z)} \|T_K(z)\|_\infty = \inf_{K(z)} \left\| \begin{bmatrix} L(z) - K(z)H(z) & -K(z) \end{bmatrix} \right\|_\infty. \tag{13}$$

Moreover, find the resulting $\gamma_{opt} \triangleq \inf_{K(z)} \|T_K(z)\|_\infty$.

There is a related noncausal H^∞ estimation problem in which the only difference is that the causality constraint on $K(z)$ is removed. In fact, the noncausal H^∞ estimation problem is very easy to solve.

Theorem 1 (Noncausal H^∞-Optimal Estimator) *The optimal H^∞ norm for the problem*

$$\min_{K(z)} \|T_K(z)\|_\infty \quad , \quad T_K(z) = \left[\begin{array}{cc} L(z) - K(z)H(z) & -K(z) \end{array} \right] \tag{14}$$

is given by

$$\gamma_s = \left\| L(z) \left[I + H^*(z^{-*})H(z) \right]^{-1} L^*(z^{-*}) \right\|_\infty^{1/2}. \tag{15}$$

Moreover, one noncausal H^∞-optimal solution is given by the Wiener smoother,

$$K_s(z) = L(z)H^*(z^{-*}) \left[I + H(z)H^*(z^{-*}) \right]^{-1}, \tag{16}$$

which has the additional property that for all $K(z)$,

$$T_{K_s}(e^{j\omega})T_{K_s}^*(e^{j\omega}) \le T_K(e^{j\omega})T_K^*(e^{j\omega}), \quad \forall \omega \in [0, 2\pi]. \tag{17}$$

Proof: Note that using a completion of squares argument we may write

$$\begin{aligned} T_K(e^{j\omega})T_K^*(e^{j\omega}) &= \left(L(e^{j\omega}) - K(e^{j\omega})H(e^{j\omega}) \right) \left(L(e^{j\omega}) - K(e^{j\omega})H(e^{j\omega}) \right)^* + K(e^{j\omega})K(e^{j\omega})^* \\ &= \left(K(e^{j\omega}) - K_s(e^{j\omega}) \right) \left(I + H(e^{j\omega})H^*(e^{j\omega}) \right) \left(K(e^{j\omega}) - K_s(e^{j\omega}) \right)^* + \\ & \quad L(e^{j\omega}) \left[I + H^*(e^{j\omega})H(e^{j\omega}) \right]^{-1} L^*(e^{j\omega}). \end{aligned}$$

Clearly, choosing $K(z) = K_s(z)$ minimizes $T_K(e^{j\omega})T_K^*(e^{j\omega})$ for <u>all</u> frequencies, thereby establishing (17). Since it also minimizes $T_K(e^{j\omega})T_K^*(e^{j\omega})$ at its *peak* frequency, $K_s(z)$ is H^∞-optimal. Finally, the expression for γ_s follows from $T_{K_s}(e^{j\omega})T_{K_s}^*(e^{j\omega}) = L(e^{j\omega}) \left[I + H^*(e^{j\omega})H(e^{j\omega}) \right]^{-1} L^*(e^{j\omega})$.

■

Thus the H^2-optimal Wiener smoother is also H^∞-optimal and, moreover, outperforms all other linear estimators (causal and noncausal) at all frequencies.

Unlike the noncausal case, closed-form expressions for the causal H^∞-optimal estimators are generally not available,[2] and one normally relaxes the optimization condition and settles for a suboptimal solution.

Problem 2 (Suboptimal H^∞ Estimation Problem) *Given a $\gamma > 0$, find a causal estimator $K(z)$ that guarantees*

$$\|T_K(z)\|_\infty = \left\| \left[\begin{array}{cc} L(z) - K(z)H(z) & -K(z) \end{array} \right] \right\|_\infty < \gamma. \tag{18}$$

This clearly requires checking whether $\gamma \ge \gamma_{opt}$.

[2] A special case for which optimal solutions can be explicitly found is the equalization problem studied in this paper.

Note also that the optimal solution, corresponding to γ_{opt}, can be found by applying a bisection method to the γ of the above Problem.

The causality requirement on $K(z)$ makes the solution to the suboptimal H^∞ estimation problem much more involved, so that additional machinery is required for the solution.

Theorem 2 (H^∞ Suboptimal Estimators) *Given $\gamma > 0$, a causal estimator $K(z)$ that achieves*

$$\left\| \begin{bmatrix} L(z) - K(z)H(z) & -K(z) \end{bmatrix} \right\|_\infty < \gamma,$$

exists if, and only if, there exists a canonical factorization of the form

$$\begin{bmatrix} I + H(z)H^*(z^{-*}) & -H(z)L^*(z^{-*}) \\ -L(z)H^*(z^{-*}) & -\gamma^2 I + L(z)L^*(z^{-*}) \end{bmatrix} = \begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} L_{11}^*(z^{-*}) & L_{21}^*(z^{-*}) \\ \mathcal{L}_{12}^*(z^{-*}) & \mathcal{L}_{22}^*(z^{-*}) \end{bmatrix}, \tag{19}$$

with $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ *causal and causally invertible, $L_{11}^{-1}(z)$ causal, and $L_{12}(z)$ strictly causal. If this is the case, then all possible H^∞ estimators of level γ are given by*

$$K(z) = \left(L_{22}(z)S(z) - L_{21}(z)\right)\left(L_{11}(z) - L_{12}(z)S(z)\right)^{-1}, \tag{20}$$

where $S(z)$ is any causal and strictly contracive transfer matrix, i.e., *$S(z)$ is causal and is such that $S(e^{j\omega})S^*(e^{j\omega}) < I$. An important choice results from taking $S(z) = 0$, so that*

$$K_{cen}(z) = -L_{21}(z)L_{11}^{-1}(z), \tag{21}$$

which is the so-called "central" filter.

Proof: The proof is detailed and is given in App. A.

■

In the above statement by $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ causally invertible, we mean that all elements of its inverse must be causal. By $L_{12}(z)$ strictly causal, we mean that it should be causal and $L_{12}(\infty) = 0$. In other words, it should have a Laurent series expansion of the form

$$L_{12}(z) = l_{12,1}z^{-1} + l_{12,2}z^{-2} + \dots.$$

The factorization (19) is known as a J-spectral factorization [14, 15], and is the indefinite extension of conventional spectral factorization. When $\gamma > \gamma_s = \left\| L(z)\left[I + H^*(z^{-*})H(z)\right]^{-1} L^*(z^{-*}) \right\|_\infty^{1/2}$, it turns out that the factorization (19), with only the requirement of having causal and causally invertible spectral factors, can generically always be performed. Moreover, the spectral factors are highly non-unique since they can be pre-multiplied by any J-unitary transfer matrix, as follows

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} \Theta(z) \quad \text{with} \quad \Theta(z) \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \Theta^*(z^{-*}) = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{22}$$

However, Theorem 2 requires more than the causal and causal invertibility properties of the spectral factor, namely, it requires that $L_{11}^{-1}(z)$ should be causal and that $L_{12}(z)$ should be strictly causal. As we shall see in the solution to the equalization problem, this is of crucial importance and is what makes the existence condition for causal filters more stringent than the existence condition for noncausal ones.

When they exist, the level-γ H^∞ estimators are highly nonunique. [Even the optimal H^∞ estimators, corresponding to γ_{opt}, are nonunique.] Eq. (20) gives a parametrization of all possible solutions. Therefore different H^∞ solutions may have varying performance with respect to other performance criteria.

The level-γ central solution (21) has further interesting properties. It maximizes the following (so-called) entropy cost

$$\epsilon_K = \int_0^{2\pi} \log \det \left[\gamma^2 I - T_K(e^{j\omega}) T_K^*(e^{j\omega})\right] d\omega, \tag{23}$$

(see *e.g.*, [16] and the proof of Theorem 2) and is therefore known as the *maximum entropy* solution. It is also *risk-sensitive optimal* [17, 18] in the following sense: Assume that the $\{u_j\}$ and $\{v_j\}$ are zero-mean, uncorrelated and temporally white Gaussian stochastic processes with unit variance, *i.e.*,

$$E \begin{bmatrix} u_i \\ v_i \end{bmatrix} \begin{bmatrix} u_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} \delta_{ij}, \tag{24}$$

then the level-γ central solution H^∞ solution minimizes

$$\gamma^2 \log \left[E \exp\left(\frac{1}{2\gamma^2}(s_i - \hat{s}_i)^*(s_i - \hat{s}_i)\right)\right]. \tag{25}$$

Thus, central H^∞ filters minimize, not the expected squared estimation error as H^2 filters do, but the expected value of the exponential of the squared estimation error [19, 20].

4 H^∞ Equalization

As mentioned earlier, the equalization problem is a special case of the general estimation problem of Sec. 3 corresponding to $L(z) = I_m$. Thus in the H^∞ approach to equalization the goal is to minimize $\left\| \begin{bmatrix} I_m - K(z)H(z) & -K(z) \end{bmatrix} \right\|_\infty$.

We should also mention that it is possible to slightly generalize the estimation problem of Sec. 3 by introducing certain weighting matrices. Thus, in the noncausal case our objective will be to find $K(z)$ to solve

$$\inf_{K(\cdot)} \sup_{u,v\in l^2, u,v,\neq 0} \frac{\|u-\hat{u}\|_Q^2}{\|u\|_Q^2+\|v\|_R^2} \triangleq \gamma_s^2, \tag{26}$$

where we have defined

$$\|u\|_Q^2 = \sum_{i=-\infty}^{\infty} u_i^* Q^{-1} u_i \quad \text{and} \quad \|v\|_R^2 = \sum_{i=-\infty}^{\infty} v_i^* R^{-1} v_i \tag{27}$$

and where $Q>0$ and $R>0$ are given weighting matrices that reflect our relative uncertainty about the unknown disturbances $\{u_i\}$ and $\{v_i\}$. Likewise, in the causal case our objective will be to find a causal $K(z)$ to solve

$$\inf_{\text{causal } K(\cdot)} \sup_{u,v\in l^2, u,v\neq 0} \frac{\|u-\hat{u}\|_Q^2}{\|u\|_Q^2+\|v\|_R^2} \triangleq \gamma_{opt}^2. \tag{28}$$

It is thus straightforward to see that, in this case, we are interested in minimizing the H^∞ norm of the transfer operator from $\{Q^{-1/2}u_i, R^{-1/2}v_i\}$ to $\{Q^{-1/2}(u_i-\hat{u}_i)\}$ which is seen to be

$$\begin{bmatrix} Q^{-1/2}\left(I_m - K(z)H(z)\right)Q^{1/2} & -Q^{-1/2}K(z)R^{1/2} \end{bmatrix} = \begin{bmatrix} I_m - Q^{-1/2}K(z)H(z)Q^{1/2} & -Q^{-1/2}K(z)R^{1/2} \end{bmatrix}. \tag{29}$$

Thus defining, $K'(z) \triangleq Q^{-1/2}K(z)R^{1/2}$ and $H'(z) \triangleq R^{-1/2}H(z)Q^{1/2}$, we see that we can rewrite this transfer matrix as

$$\begin{bmatrix} I_m - K'(z)H'(z) & -K'(z) \end{bmatrix}. \tag{30}$$

Thus there is no loss of generality in assuming $Q = I_m$ and $R = I_p$, and we shall do so throughout.

We should also remark that one can give the following inequality for γ_s and γ_{opt}:

$$\gamma_s \leq \gamma_{opt} \leq 1. \tag{31}$$

The inequality $\gamma_s \leq \gamma_{opt}$ is quite clear, since noncausal equalizers have access to more information than causal ones and cannot therefore have worse performance. The second inequality follows from the fact that if we perform no equalization, *i.e.*, $K(z)=0$, then we have $\{\hat{u}_i\}=\{0\}$, and we may write

$$\frac{\|u-\hat{u}\|_Q^2}{\|u\|_Q^2+\|v\|_R^2} = \frac{\|u\|_Q^2}{\|u\|_Q^2+\|v\|_R^2} \leq 1.$$

Thus $\gamma = 1$ is the worst-case energy gain obtained from doing no equalization.

Therefore how much the optimal values of γ_s and γ_{opt} can be reduced from unity shows how successful we are in the equalization problem.

4.1 The Noncausal Case

The solution in this case is a special case of Theorem 1 and is given below.

Theorem 3 (Noncausal H^∞-Optimal Equalizer) *The optimal H^∞ norm for the problem*

$$\min_{K(z)} \left\| \begin{bmatrix} I_m - K(z)H(z) & -K(z) \end{bmatrix} \right\|_\infty \tag{32}$$

is given by

$$\gamma_s^2 = \sup_{\omega\in[0,2\pi]} \bar{\sigma}\left[I + H^*(e^{j\omega})H(e^{j\omega})\right]^{-1}. \tag{33}$$

Moreover, one noncausal H^∞-optimal solution is given by the noncausal MMSE equalizer,

$$K_s(z) = H^*(z^{-*})\left[I + H(z)H^*(z^{-*})\right]^{-1}, \tag{34}$$

which has the additional property that for all $K(z)$.

$$T_{K_s}(e^{j\omega})T_{K_s}^*(e^{j\omega}) \le T_K(e^{j\omega})T_K^*(e^{j\omega}), \quad \forall\omega \in [0, 2\pi]. \tag{35}$$

Remarks:

(i) Focusing on the scalar case is instructive. In this case, we have

$$\gamma_s^2 = \sup_{\omega\in[0,2\pi]} \frac{1}{1 + |H(e^{j\omega})|^2}. \tag{36}$$

In other words, γ_s is achieved at those frequencies for which $\left|H(e^{j\omega})\right|^2$ takes its smallest value.

(ii) Consider now the general case where $H(z)$ is a $p \times m$ transfer matrix. If $m > p$ (so that there are more signals to estimate than to observe), we have

$$\gamma_s = 1, \tag{37}$$

since $H^*(z^{-*})H(z)$ will be rank deficient. More importantly, in the case of a square channel ($p = m$), if the channel has a unit circle zero then $\gamma_s = 1$ (since $H^*(e^{j\omega})H(e^{j\omega})$ will be rank deficient at the frequency where the zero occurs).

(iii) Recall from our earlier discusions that $\gamma_s = 1$ implies that we have no improvement over not equalizing at all ($K(z) = 0$). This is quite clear when we have a unit circle zero, corresponding to frequency ω_1, say. In this case the output of the channel cannot contain sinusoids of frequency ω_1, and hence if $\{u_i\}$ is precisely such a sinusoid, then we cannot estimate it since we cannot observe it at the output of $H(z)$.

5 Causal H^∞ Equalization

Our main results are concerned with the case where the equalizer is restricted to being causal.

5.1 The Square Case

In this section we shall assume that $p = m$, *i.e.*, that the number of input and observations signals coincide. We shall presently show that, in this case, we can find an explicit expression for the factorization (19) of Theorem 2. This is what allows us to, contrary to general H^∞ estimation problems, obtain a closed-form solution to the causal H^∞ equalization problem.

5.1.1 J-Spectral Factorization in the Square Case

Note that in the equalization problem the transfer matrix whose J-spectral factorization we are seeking is given by,

$$\begin{bmatrix} I + H(z)H^*(z^{-*}) & -H(z) \\ -H^*(z^{-*}) & -\gamma^2 I + I \end{bmatrix}. \tag{38}$$

[We have taken $L(z) = I_m$ in (19).] Since we know $\gamma \geq 1$ is achievable, let us consider (38) for a given $\gamma_s < \gamma < 1$. Once γ is strictly less than one we may perform the following block upper-diagonal-lower (UDL) factorization,

$$\begin{bmatrix} I + H(z)H^*(z^{-*}) & -H(z) \\ -H^*(z^{-*}) & -\gamma^2 I + I \end{bmatrix} =$$

$$\begin{bmatrix} I & -\frac{H(z)}{1-\gamma^2} \\ 0 & I \end{bmatrix} \begin{bmatrix} \underbrace{I + H(z)H^*(z^{-*}) - \frac{H(z)H^*(z^{-*})}{1-\gamma^2}}_{I - \frac{\gamma^2}{1-\gamma^2}H(z)H^*(z^{-*})} & 0 \\ 0 & I - \gamma^2 I \end{bmatrix} \begin{bmatrix} I & 0 \\ -\frac{H^*(z^{-*})}{1-\gamma^2} & I \end{bmatrix}. \tag{39}$$

Now since $\gamma > \gamma_s$, we have

$$\begin{aligned} & \gamma^2 I > \left[I + H^*(e^{j\omega})H(e^{j\omega})\right]^{-1} \\ \Rightarrow \quad & \gamma^{-2} I < I + H^*(e^{j\omega})H(e^{j\omega}) \\ \Rightarrow \quad & (\gamma^{-2} - 1)I < H^*(e^{j\omega})H(e^{j\omega}) \\ \Rightarrow \quad & (\gamma^{-2} - 1)H^{-*}(e^{j\omega})H^{-1}(e^{j\omega}) < I \\ \Rightarrow \quad & \frac{\gamma^2}{1-\gamma^2}H(e^{j\omega})H^*(e^{j\omega}) > I, \end{aligned}$$

where in the fourth step we have used our assumption that $\gamma_s < 1$, so that $H(e^{j\omega})$ is full rank for all ω.

The last inequality implies that we can introduce the canonical spectral factorization

$$\Delta(z)R_\Delta\Delta^*(z^{-*}) = \frac{\gamma^2}{1-\gamma^2}H(z)H^*(z^{-*}) - I > 0, \tag{40}$$

with $\Delta(z)$ monic ($\Delta(\infty) = I$) and causal and causally invertible, and $R_\Delta > 0$. With this definition we can write the factorization (39) as

$$\begin{aligned}
\begin{bmatrix} I + H(z)H^*(z^{-*}) & -H(z) \\ -H^*(z^{-*}) & -\gamma^2 I + I \end{bmatrix} &= \begin{bmatrix} \Delta(z)R_\Delta^{1/2} & -\frac{H(z)}{\sqrt{1-\gamma^2}} \\ 0 & \sqrt{1-\gamma^2}I \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} R_\Delta^{*/2}\Delta^*(z^{-*}) & 0 \\ -\frac{H^*(z^{-*})}{\sqrt{1-\gamma^2}} & \sqrt{1-\gamma^2}I \end{bmatrix} \\
&= \begin{bmatrix} -\frac{H(z)}{\sqrt{1-\gamma^2}} & \Delta(z)R_\Delta^{1/2} \\ \sqrt{1-\gamma^2}I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} -\frac{H^*(z^{-*})}{\sqrt{1-\gamma^2}} & \sqrt{1-\gamma^2}I \\ R_\Delta^{*/2}\Delta^*(z^{-*}) & 0 \end{bmatrix}.
\end{aligned}$$

Note that the above factorization is a J-spectral factorization since the spectral factor

$$P(z) \triangleq \begin{bmatrix} -\frac{H(z)}{\sqrt{1-\gamma^2}} & \Delta(z)R_\Delta^{1/2} \\ \sqrt{1-\gamma^2}I & 0 \end{bmatrix}, \tag{41}$$

is causal, and causally invertible, since

$$P^{-1}(z) = \begin{bmatrix} 0 & \frac{1}{\sqrt{1-\gamma^2}}I \\ R_\Delta^{-1/2}\Delta^{-1}(z) & \frac{R_\Delta^{-1/2}\Delta^{-1}(z)H(z)}{1-\gamma^2} \end{bmatrix}. \tag{42}$$

is causal. However, it is not quite the desired factorization of Theorem 2 since the block $(1,2)$ entry, $P_{12}(z)$, is not strictly causal. To transform $P(z)$ into a transfer matrix such that the $(1,2)$ entry is strictly causal, we must premultiply $P(z)$ by an appropriate constant[3] J-unitary matrix, Θ. In fact, we need only concentrate on $P(\infty)$. Thus, we must have

$$P(\infty)\Theta = \begin{bmatrix} -\frac{h_0}{\sqrt{1-\gamma^2}} & R_\Delta^{1/2} \\ \sqrt{1-\gamma^2}I & 0 \end{bmatrix} \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}. \tag{43}$$

Since Θ is J-unitary we have

$$P(\infty)JP^*(\infty) = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} J \begin{bmatrix} A^* & B^* \\ 0 & C^* \end{bmatrix}, \quad J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{44}$$

Comparing the $(1,1)$ block entries in the above equality we have

$$\frac{h_0 h_0^*}{1-\gamma^2} - R_\Delta = AA^*, \tag{45}$$

which implies that $A = \left(\frac{h_0 h_0^*}{1-\gamma^2} - R_\Delta\right)^{1/2}$ and that $P_{12}(z)$ can be made strictly causal only if,

$$\frac{h_0 h_0^*}{1-\gamma^2} - R_\Delta > 0. \tag{46}$$

[3]The J-unitary matrix must be constant so as not to destroy the causal and causal invertibility property of $P(z)$.

Comparing the $(2,1)$ block entries of (44) yields

$$-h_0^* = BA^*, \tag{47}$$

so that $B = -h_0^* A^{-*}$. Finally, comparing the $(2,2)$ block entries yields

$$(1-\gamma^2)I = BB^* - CC^*, \tag{48}$$

or,

$$\begin{aligned} CC^* &= -(1-\gamma^2)I + h_0^* \left(\frac{h_0 h_0^*}{1-\gamma^2} - R_\Delta \right)^{-1} h_0 \\ &= (1-\gamma^2)^2 \left[h_0^* R_\Delta^{-1} h_0 - (1-\gamma^2)I \right]^{-1}, \end{aligned}$$

so that $C = (1-\gamma^2)\left[h_0^* R_\Delta^{-1} h_0 - (1-\gamma^2)I\right]^{-*/2}$. We should also remark that (46) guarantees $h_0^* R_\Delta^{-1} h_0 - (1-\gamma^2)I > 0$ so that C is well defined. Indeed, two different block lower-diagonal-upper and block upper-diagonal-lower factorizations of the matrix

$$N = \begin{bmatrix} (1-\gamma^2)I & h_0^* \\ h_0 & R_\Delta \end{bmatrix},$$

show that the matrices

$$\begin{bmatrix} (1-\gamma^2)I & 0 \\ 0 & R_\Delta - \frac{h_0 h_0^*}{1-\gamma^2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (1-\gamma^2)I - h_0^* R_\Delta^{-1} h_0 & 0 \\ 0 & R_\Delta \end{bmatrix}$$

are congruent and therefore must have the same inertia. Since $(1-\gamma^2)I > 0$ and $R_\Delta - \frac{h_0 h_0^*}{1-\gamma^2} < 0$, we conclude that N has m positive eigenvalues and m negative eigenvalues. Moreover, since $R_\Delta > 0$, we conclude that $(1-\gamma^2)I - h_0^* R_\Delta^{-1} h_0 < 0$, which is the desired result.

Having found the elements $\{A, B, C\}$, we can now find the desired J-unitary matrix,

$$\begin{aligned} \Theta &= \begin{bmatrix} -\frac{h_0}{\sqrt{1-\gamma^2}} & R_\Delta^{1/2} \\ \sqrt{1-\gamma^2}I & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\frac{1}{\sqrt{1-\gamma^2}} I \\ R_\Delta^{-1/2} & \frac{R_\Delta^{-1/2} h_0}{1-\gamma^2} \end{bmatrix} \begin{bmatrix} A & 0 \\ -h_0^* A^{-*} & C \end{bmatrix}, \end{aligned}$$

and thereby the desired canonical factor

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} \Theta. \tag{49}$$

The results of this section are summarized in the following lemma.

Lemma 1 (J-Spectral Factorization) *Suppose the $m \times m$ causal matrix function $H(z) = h_0 + h_1 z^{-1} + \dots$ has full rank on $|z| = 1$, and let γ be such that*

$$1 > \gamma^2 > \sup_{\omega \in [0,2\pi]} \bar{\sigma} \left[I + H^*(e^{j\omega}) H(e^{j\omega}) \right]^{-1}. \tag{50}$$

Then a factorization of the form

$$\begin{bmatrix} I + H(z)H^*(z^{-*}) & -H(z) \\ H^*(z^{-*}) & -\gamma^2 I + I \end{bmatrix} = \begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} L_{11}^*(z^{-*}) & L_{21}^*(z^{-*}) \\ \mathcal{L}_{12}^*(z^{-*}) & \mathcal{L}_{22}^*(z^{-*}) \end{bmatrix}, \tag{51}$$

with $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ *causal and causally invertible, and $L_{12}(z)$ strictly causal, exists if, and only if,*

$$\frac{h_0 h_0^*}{1 - \gamma^2} - R_\Delta > 0, \tag{52}$$

where R_Δ is found from the canonical spectral factorization

$$\Delta(z) R_\Delta \Delta^*(z^{-*}) = \frac{\gamma^2}{1 - \gamma^2} H(z) H^*(z^{-*}) - I > 0, \tag{53}$$

with $\Delta(z)$ monic and causal and causally invertible. If this is the case, then we have

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \begin{bmatrix} \left[H(z) h_0^* - (1 - \gamma^2) \Delta(z) R_\Delta \right] \frac{A^{-*}}{1-\gamma^2} & \frac{-H(z) + \Delta(z) h_0}{1-\gamma^2} C \\ -h_0^* A^{-*} & C \end{bmatrix}, \tag{54}$$

where A and C are found the factorizations

$$AA^* = \frac{h_0 h_0^*}{1 - \gamma^2} - R_\Delta \quad \textit{and} \quad CC^* = (1 - \gamma^2)^2 \left[h_0^* R_\Delta^{-1} h_0 - (1 - \gamma^2) I \right]^{-1}. \tag{55}$$

From the above lemma, and using Theorem 2, it follows that a level-γ H^∞ equalizer exists if, and only if, (52) is satisfied and

$$L_{11}^{-1}(z) = (1 - \gamma^2) A^* \left[H(z) h_0^* - (1 - \gamma^2) \Delta(z) R_\Delta \right]^{-1} \quad \text{is causal.} \tag{56}$$

Thus our problem is reduced to checking (52) and (56). This may appear to be a formidable task, but we shall presently see that the answer depends upon whether $H(z)$ is minimum phase or not.

5.1.2 The Minimum Phase Case

Let us first assume that the $m \times m$ causal matrix function $H(z)$ is minimum phase, *i.e.*, that its inverse $H^{-1}(z)$ is causal. Then we have the following result.

Lemma 2 *Consider the setting of Lemma 1 and suppose that $H(z)$ is minimum phase. Then for all γ satisfying (50) we have*

$$\frac{h_0 h_0^*}{1-\gamma^2} - R_\Delta > 0. \tag{57}$$

Proof: Note that from (53),

$$\Delta(e^{j\omega}) R_\Delta \Delta^*(e^{j\omega}) = \frac{\gamma^2}{1-\gamma^2} H(e^{j\omega}) H^*(e^{j\omega}) - 1.$$

Thus,

$$\begin{aligned}
& (1-\gamma^2)\Delta(e^{j\omega}) R_\Delta \Delta^*(e^{j\omega}) = \gamma^2 H(e^{j\omega})H^*(e^{j\omega}) - (1-\gamma^2) < H(e^{j\omega})H^*(e^{j\omega}) \\
\Rightarrow \quad & H^{-1}(e^{j\omega})\Delta(e^{j\omega})(1-\gamma^2) R_\Delta \Delta^*(e^{j\omega}) H^{-*}(e^{j\omega}) < I \\
\Rightarrow \quad & H^{-1}(e^{j\omega})\Delta(e^{j\omega}) h_0 h_0^{-1}(1-\gamma^2) R_\Delta h_0^{-*} h_0^* \Delta^*(e^{j\omega}) H^{-*}(e^{j\omega}) < I.
\end{aligned} \tag{58}$$

Now the transfer matrix $M(z) \stackrel{\Delta}{=} H^{-1}(z)\Delta(z)h_0$ is causal (since $H^{-1}(z)$ and $\Delta(z)$ are causal) and monic (since $M(\infty) = H^{-1}(\infty)\Delta(\infty)h_0 = h_0^{-1}\cdot I \cdot h_0 = I$), and we may write

$$M(z) = I + m_1 z^{-1} + m_2 z^{-2} + \ldots.$$

Therefore

$$\begin{aligned}
& \left(I + m_1 e^{-j\omega} + \ldots\right) h_0^{-1}(1-\gamma^2) R_\Delta h_0^{-*} \left(I + m_1 e^{-j\omega} + \ldots\right)^* < I \\
\Rightarrow \quad & \frac{1}{2\pi}\int_0^{2\pi} \left(I + m_1 e^{-j\omega} + \ldots\right) h_0^{-1}(1-\gamma^2) R_\Delta h_0^{-*} \left(I + m_1 e^{-j\omega} + \ldots\right)^* d\omega < \frac{1}{2\pi}\int_0^{2\pi} I d\omega \\
\Rightarrow \quad & h_0^{-1}(1-\gamma^2) R_\Delta h_0^{-*} + m_1 h_0^{-1}(1-\gamma^2) R_\Delta h_0^{-*} m_1^* + \ldots < I \\
\Rightarrow \quad & h_0^{-1}(1-\gamma^2) R_\Delta h_0^{-*} < I \\
\Rightarrow \quad & (1-\gamma^2) R_\Delta < h_0 h_0^*.
\end{aligned}$$

■

The next result establishes the causal invertibility of $L_{11}(z)$.

Lemma 3 *Consider the setting of Lemma 1 and suppose that $H(z)$ is minimum phase. Then for all γ satisfying (50) the transfer matrix*

$$L_{11}^{-1}(z) = (1-\gamma^2) A^* \left[H(z)h_0^* - (1-\gamma^2)\Delta(z) R_\Delta\right]^{-1}, \tag{59}$$

is causal.

Proof: Let us first write

$$L_{11}^{-1}(z) = (1-\gamma^2)A^* h_0^{-*}\left[I - (1-\gamma^2)H^{-1}(z)\Delta(z)R_\Delta h_0^{-*}\right]^{-1} H^{-1}(z),$$

where we have used the fact that h_0 is invertible since $H(z)$ is minimum phase (if h_0 is singular, then $H(z)$ will have a zero at $z = \infty$). Now since $H^{-1}(z)$ is causal, $L_{11}^{-1}(z)$ will be causal if $\left[I - (1-\gamma^2)H^{-1}(z)\Delta(z)R_\Delta h_0^{-*}\right]^{-1}$ is causal. We shall presently show this by appealing to Lemma 9 in App.B, and showing that

$$A(z) \triangleq (1-\gamma^2)H^{-1}(z)\Delta(z)R_\Delta h_0^{-*}, \tag{60}$$

is a strict contraction. Indeed,

$$\begin{aligned} A(e^{j\omega})A^*(e^{j\omega}) &= (1-\gamma^2)H^{-1}(e^{j\omega})\Delta(e^{j\omega})R_\Delta \underbrace{h_0^{-*}h_0^{-1}}_{<\frac{1}{1-\gamma^2}R_\Delta^{-1}} R_\Delta\Delta^*(e^{j\omega})H^{-*}(e^{j\omega})(1-\gamma^2) \\ &< H^{-1}(e^{j\omega})\Delta(e^{j\omega})(1-\gamma^2)R_\Delta\Delta^*(e^{j\omega})H^{-*}(e^{j\omega}) \\ &< I, \end{aligned}$$

where in the second step we used $h_0 h_0^* > (1-\gamma^2)R_\Delta$ so that $h_0^{-*}h_0^{-1} < \frac{1}{1-\gamma^2}R_\Delta^{-1}$, and where in the last step we used (58).

■

The above lemmas show that, in the minimum phase case, whenever a J-spectral factorization exists, then $L_{12}(z)$ can be made strictly causal and $L_{11}(z)$ can be made causally invertible. This essentially means that in the minimum phase case, we have $\gamma_{opt} = \gamma_s$.

5.1.3 The Non-Minimum Phase Case

We now assume that the $m \times m$ causal matrix function $H(z)$ is non-minimum phase, *i.e.*, that its inverse $H^{-1}(z)$ is noncausal. Here we have the following result.

Lemma 4 *Consider the setting of Lemma 1 and suppose that $H(z)$ is non-minimum phase, and that*

$$\frac{h_0 h_0^*}{1-\gamma^2} - R_\Delta > 0. \tag{61}$$

Then the transfer matrix

$$L_{11}^{-1}(z) = (1-\gamma^2)A^*\left[H(z)h_0^* - (1-\gamma^2)\Delta(z)R_\Delta\right]^{-1}, \tag{62}$$

is noncausal.

Proof: First note that we may write

$$L_{11}^{-1}(z) = -A^* R_\Delta^{-*/2} \left[I - R_\Delta^{-1/2} \Delta^{-1}(z) H(z) h_0^* R_\Delta^{-*/2} \right]^{-1} R_\Delta^{-1/2} \Delta^{-1}(z).$$

Now when (61) holds, using an argument similar to the one presented in the proof of Lemma 3, one can show that the causal transfer matrix,

$$C(z) \stackrel{\Delta}{=} R_\Delta^{-1/2} \Delta^{-1}(z) H(z) h_0^* R_\Delta^{-*/2}, \tag{63}$$

is a strict expansion, *i.e.*,

$$C(e^{j\omega}) C(e^{j\omega}) > I, \quad \forall \omega.$$

Note moreover, that since, $H^{-1}(z)$ is noncausal, and

$$H^{-1}(z) = h_0^* R_\Delta^{-*/2} C^{-1}(z) R_\Delta^{-1/2} \Delta^{-1}(z),$$

the transfer matrix $C^{-1}(z)$ is noncausal as well. (Otherwise, the noncausal $H^{-1}(z)$ would be the product of causal transfer matrices). Lemma 10 now implies that, since $C(z)$ is a causal expansion and $C^{-1}(z)$ is noncausal, the transfer matrix $(I - C(z))^{-1}$ must be noncausal. This in turn implies that $L_{11}^{-1}(z)$ is noncausal, since

$$(I - C(z))^{-1} = -R_\Delta^{*/2} A^{-*} L_{11}^{-1}(z) \Delta(z) R_\Delta^{1/2},$$

cannot be the product of causal transfer matrices.

■

Recall that for a level-γ H^∞ equalizer, with $\gamma < 1$, to exist we require both conditions (52) and (56). What we have just shown is that, when $H(z)$ is non-minimum phase, even if (52) is satisfied (56) is not. This essentially means that in the non-minimum phase case we have,

$$\gamma_{opt} = 1, \tag{64}$$

since $\gamma = 1$ is always achievable by $K(z) = 0$.

5.1.4 Main Result

We are now in a position to the state the main result of this paper.

Theorem 4 (H^∞ Equalization in the Square Case) *Consider the $m \times m$ causal transfer matrix $H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots$, and suppose that we want to solve the problem*

$$\min_{\text{causal } K(\cdot)} \left\| \begin{bmatrix} I - K(z)H(z) & -K(z) \end{bmatrix} \right\|_\infty \stackrel{\Delta}{=} \gamma_{opt}.$$

(i) If $H(z)$ is minimum phase, i.e., *if $H^{-1}(z)$ is analytic on $|z| \geq 1$, then the minimax energy gain is given by*

$$\gamma_{opt}^2 = \sup_{\omega \in [0,2\pi]} \bar{\sigma}\left[I + H^*(e^{j\omega})H(e^{j\omega})\right]^{-1}. \tag{65}$$

Moreover, for any $\gamma > \gamma_{opt}$, all causal equalizers that guarantee $\left\|\begin{bmatrix} I - K(z)H(z) & -K(z) \end{bmatrix}\right\|_\infty^2 \leq \gamma^2$ are given by

$$K(z) = \left(L_{22}(z)S(z) - L_{21}(z)\right)\left(L_{11}(z) - L_{12}(z)S(z)\right)^{-1}, \tag{66}$$

where the $L_{ij}(z)$ are given by

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \begin{bmatrix} \left[H(z)h_0^* - (1-\gamma^2)\Delta(z)R_\Delta\right]\frac{A^{-*}}{1-\gamma^2} & \frac{-H(z)+\Delta(z)h_0}{1-\gamma^2}C \\ -h_0^*A^{-*} & C \end{bmatrix}, \tag{67}$$

with the monic and minimum phase transfer matrix $\Delta(z)$ and the matrix R_Δ found from the canonical spectral factorization,

$$\Delta(z)R_\Delta\Delta^*(z^{-*}) = \frac{\gamma^2}{1-\gamma^2}H(z)H^*(z^{-*}) - I > 0, \tag{68}$$

where the matrices $\{A, C\}$ are found from the factorizations,

$$AA^* = \frac{h_0h_0^*}{1-\gamma^2} - R_\Delta \quad \text{and} \quad CC^* = (1-\gamma^2)^2\left[h_0^*R_\Delta^{-1}h_0 - (1-\gamma^2)I\right]^{-1} \tag{69}$$

and where $S(z)$ is any causal contraction contraction, i.e., *$S(z)$ is analytic on $|z| \geq 1$ and*

$$S^*(z^{-*})S(z) \leq I, \quad \forall\ |z| = 1. \tag{70}$$

(ii) If $H(z)$ is non-minimum phase, i.e., *if $H^{-1}(z)$ is not analytic on $|z| \geq 1$, then the minimax energy gain is given by*

$$\gamma_{opt} = 1. \tag{71}$$

Remarks:

(i) Note that when $H(z)$ is minimum phase, $\gamma_{opt} = \gamma_s$! This implies that for minimum phase channels causal equalizers perform as well as noncausal ones, and that (from an H^∞ point of view) there is no gain in knowing future values of the observations signal $\{y_i\}$.

(ii) Recall from Theorem 3, that the central noncausal equalizer at least outperforms all other equalizers (causal or noncausal) at all frequencies. Theorem 4 therefore states that if $H(z)$ is minimum phase, then the central noncausal equalizer cannot outperform the best causal equalizer in terms of its worst-case performance (which occurs at certain frequencies).

(iii) However, if $H(z)$ is non-minimum phase, then $\gamma_{opt} = 1$! Thus causal equalization of non-minimum phase channels is not possible, since $\gamma = 1$ is the same bound obtained by not equalizing at all ($K(z) = 0$).

(iv) Suppose now that $H(z)$ is rational. Then if all the zeros of $H(z)$ lie strictly inside the unit circle, causal equalizers perform as good as noncausal ones. As soon as a zero lies on the unit circle then equalization is not possible ($\gamma_s = \gamma_{opt} = 1$) since the channel cannot pass certain frequencies. However, as soon as a zero crosses the unit circle noncausal equalizers can now perform whereas causal ones cannot. There is therefore a clear transition occuring at the unit circle and a strict dichotomy between minimum phase and non-minimum phase channels as far as causal equalization is concerned.

(v) Some understanding of the result of Theorem 4 for non-minimum phase channels can be obtained by considering the simple (perhaps the simplest) non-minimum phase channel $H(z) = z^{-1}$ (a pure delay). In this case it is quite clear that causal equalization is not possible, since at time i the equalizer has access only to noisy observations of $\{u_j\}_{j=-\infty}^{i-1}$ and has no knowledge of u_i in order to be able to estimate it.

(vi) A similar behaviour to Theorem 4 can be observed had we studied the equalization problem from an H^2 point of view, though the result is not as pronounced. In the scalar case (which for simplicity we shall only consider) the H^2 norm of the H^2-optimal causal equalizer is given by

$$d_2 = 1 - \frac{|h_0|^2}{R_e}, \tag{72}$$

where R_e is found from the spectral factorization

$$1 + H(z)H^*(z^{-*}) = M^*(z^{-*})R_e M(z), \tag{73}$$

with $M(z)$ monic and minimum phase. It is now easy to show that for all channels, $H_1(z)$ and $H_2(z)$, that have the same spectrum (*i.e.*, $1 + H_1(z)H_1^*(z^{-*}) = 1 + H_2(z)H_2^*(z^{-*})$), so that noncausal equalizers have the same performance, $|h_0|^2$ is largest for the minimum phase channel. Thus d_2 is smallest for the minimum phase channel which means that the corresponding equalizer has the best H^2 performance. (In fact, it can also be shown that d_2 increases as the number of non-minimum phase zeros increases.)

5.1.5 Special Cases

Theorem 4 gives a full parametrization of all possible level-γ H^∞ equalizers in terms of a causal strict contraction, $S(z)$. The most natural choice is $S(z) = 0$, which corresponds to the central equalizer

$$K_{cen}(z) = (1-\gamma^2)h_0^* \left[H(z)h_0^* - (1-\gamma^2)\Delta(z)R_\Delta\right]^{-1}. \tag{74}$$

As mentioned earlier, the central equalizer has various other desirable optimality properties, such as being risk-sensitive optimal and being the maximum entropy solution. We shall not go into the details here, and shall just mention that the risk-sensitive optimality property of minimizing

$$E \exp\left[\frac{1}{2\gamma^2}(u_i - \hat{u}_i)^*(u_i - \hat{u}_i)\right], \tag{75}$$

may be useful for digital communications since exponentially larger penalties are being applied to larger estimation errors, which is typically where detection errors (detecting a 0 as a 1, say) occur. This observation is studied in more detail in [21] with mixed results.

A less obvious, but nonetheless intriguing, choice is

$$S(z) = -(1-\gamma^2)C^{-1}h_0^{-1}R_\Delta A^{-*}, \tag{76}$$

a constant matrix. Indeed $S(z)$ is a strict contraction since

$$\begin{aligned}
CC^* &= (1-\gamma^2)^2\left[h_0^* R_\Delta^{-1} h_0 - (1-\gamma^2)I\right]^{-1} \\
&= (1-\gamma^2)^2\left[h_0^{-1}R_\Delta h_0^{-*} - h_0^{-1}R_\Delta h_0^{-*}\left[h_0^{-1}R_\Delta h_0^{-*} - \frac{1}{1-\gamma^2}I\right]^{-1} h_0^{-1}R_\Delta h_0^{-*}\right] \\
&= (1-\gamma^2)^2\left[h_0^{-1}R_\Delta h_0^{-*} - h_0^{-1}R_\Delta\left[R_\Delta - \frac{h_0 h_0^*}{1-\gamma^2}\right]^{-1} R_\Delta h_0^{-*}\right] \\
&= (1-\gamma^2)^2 h_0^{-1}R_\Delta h_0^{-*} + (1-\gamma^2)^2 h_0^{-1}R_\Delta A^{-*}A^{-1}R_\Delta h_0^{-*} \\
&< (1-\gamma^2)^2 h_0^{-1}R_\Delta A^{-*}A^{-1}R_\Delta h_0^{-*},
\end{aligned}$$

so that

$$SS^* = (1-\gamma^2)C^{-1}h_0^{-1}R_\Delta A^{-*}A^{-1}R_\Delta h_0^{-*}C^{-*}(1-\gamma^2) < I.$$

With this choice of $S(z)$, Eq. (66) shows that we obtain

$$K(z) = (1-\gamma^2)H^{-1}(z), \tag{77}$$

i.e., one H^∞ optimal equalizer is simply a scaled version of the inverse of the channel! In other words suitably scaling the zero-forcing equalizer is H^∞-optimal.

Lemma 5 (Zero-Forcing Equalizer) *For any γ such that*

$$1 > \gamma^2 > \sup_{\omega\in[0,2\pi]} \bar{\sigma}\left[I + H^*(e^{j\omega})H(e^{j\omega})\right]^{-1}, \tag{78}$$

(Eq. (78) implies that $H(z)$ is minimum phase) the equalizer $K(z) = (1-\gamma^2)H^{-1}(z)$ achieves

$$\|T_K(z)\|_\infty = \left\| \begin{bmatrix} I - K(z)H(z) & -K(z) \end{bmatrix} \right\|_\infty < \gamma. \tag{79}$$

Proof: We have already presented a proof in the arguments leading to the statement of the lemma. A more direct proof is also possible. Note that with $K(z) = (1-\gamma^2)H^{-1}(z)$, we have

$$T_K(z) = \begin{bmatrix} \gamma^2 I & -(1-\gamma^2)H^{-1}(z) \end{bmatrix}.$$

Therefore

$$\begin{aligned} T_K(e^{j\omega})T_K^*(e^{j\omega}) &= \gamma^4 I + (1-\gamma^2)^2 H^{-1}(e^{j\omega})H^{-*}(e^{j\omega}) \\ &< \gamma^4 I + (1-\gamma^2)^2 \cdot \frac{\gamma^2}{1-\gamma^2} I = \gamma^2 I, \end{aligned}$$

where in the second step we have used $H^*(e^{j\omega})H(e^{j\omega}) > \frac{1-\gamma^2}{\gamma^2} I$, so that $H^{-1}(e^{j\omega})H^{-*}(e^{j\omega}) < \frac{\gamma^2}{1-\gamma^2} I$.

∎

Note that if there were no additive disturbance present ($\{v_i\} = \{0\}$), the inverse of the channel would perfectly reconstruct the input signal. The scaling factor $1-\gamma^2$, though, is crucial here — if $H(z)$ has zeros close to the unit circle the frequency response of $H(z)$ will be very large at certain frequencies, resulting in large amplification of the noise at those frequencies. However, in this case γ_{opt} will be close to one, and thus the factor $1-\gamma^2$ prohibits such large amplifications.

Nonetheless, even though it is H^∞-optimal, scaling and inverting the plant is not a very satisfactory solution since it has poor average performance and leads to flat error spectra.

To illustrate this fact, and to compare the various equalizers considered so far, let us consider the problem of equalizing a scalar minimum phase channel, $H(z)$. In Figs. 3-5, we have plotted the error spectra $|T_K(e^{j\omega})|^2$, corresponding to the noncausal H^2 (or noncausal central H^∞) optimal equalizer, the causal H^2-optimal equalizer, the causal central H^∞-optimal equalizer, and the causal H^∞-optimal (scaled) zero-forcing equalizer, for several different minimum phase channels. Reviewing these figures should allow the reader to gain some perspective and "feel" for the relative performances of these different equalizers.

As can be seen, the H^2 and H^∞ optimal noncausal equalizer outperforms all other equalizers at all frequencies. Its error spectrum thus serves as a lower bound for the error spectra of all other equalizers. The H^2-optimal equalizer has the best average performance (under the appropriate stochastic

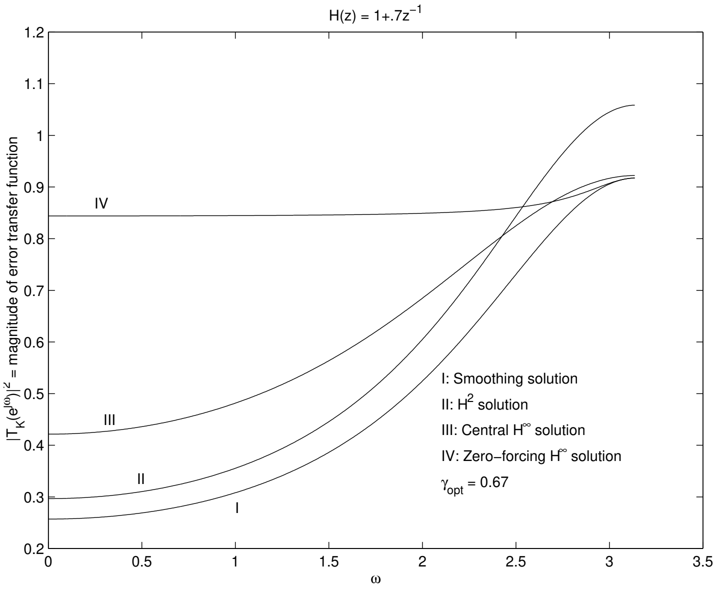

Figure 3: *Inversion of the noisy channel* $H(z) = 1 + .7z^{-1}$.

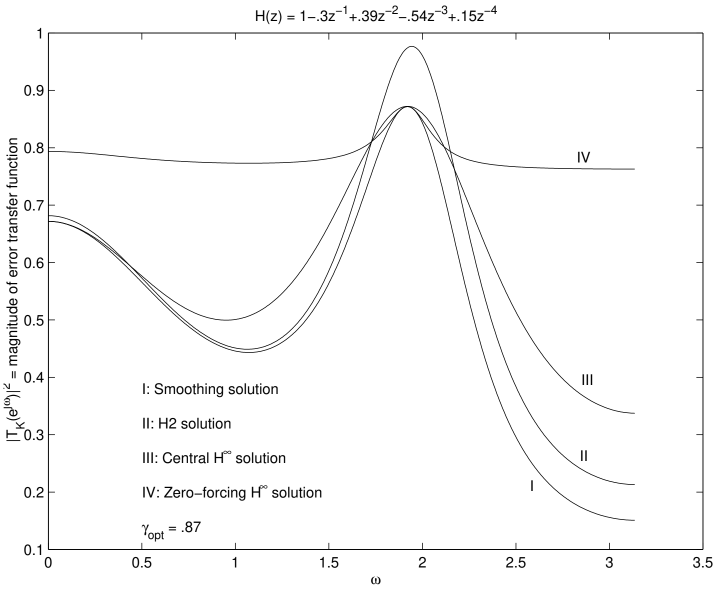

Figure 4: *Inversion of the noisy channel* $H(z) = 1 - .3z^{-1} + .39z^{-2} - .54z^{-3} + .15z^{-4}$.

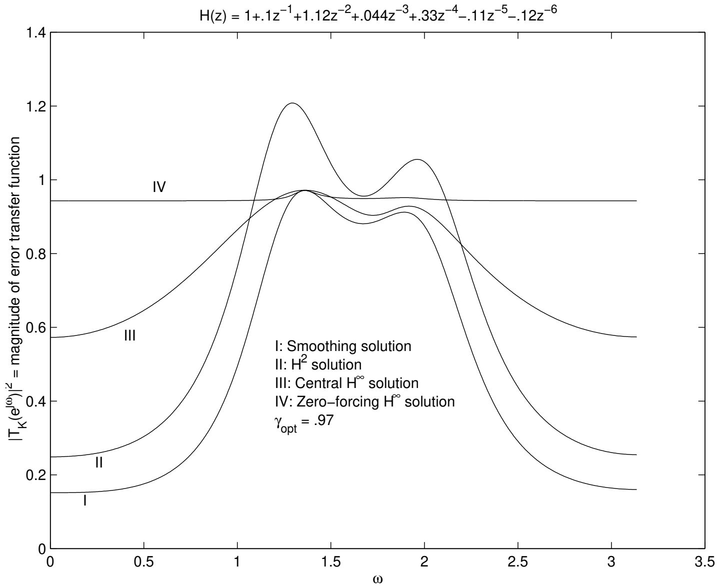

Figure 5: *Inversion of the noisy channel* $H(z) = 1 + .1z^{-1} + 1.12z^{-2} + .044z^{-3} + .33z^{-4} - .11z^{-5} - .12z^{-6}$.

assumptions), and hence has the smallest area under the spectrum curve, among all causal estimators. The (central) H^∞-optimal and the zero-forcing H^∞-optimal equalizers have the best worst-case performance, and hence their spectra has the smallest peak, over all causal estimators. Since the channels are minimum phase, these peaks coincide with the peak of the noncausal H^∞-optimal equalizer.

These figures also indicate the tradeoffs between H^2 and H^∞ equalizers. Although H^2-optimal equalizers have the smallest area under the spectrum curve, they tend to have large peaks which correspond to nonrobust behaviour at certain frequencies. On the other hand, although the H^∞-optimal estimators are robust, and have the smallest possible peaks, they have larger areas under their spectrum curve and *may* have poor average performance. This is particularly true of the scaled zero-forcing equalizer which has a very "flat" spectrum.

5.2 The Non-Square Case

So far we have only considered the case of a square channel ($p = m$). When $p \neq m$, we have the following result.

Theorem 5 (Causal H^∞ Equalizer for Nonsquare Plant) *Consider the $p \times m$ causal transfer matrix $H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots$, where $p \neq m$, and where we want to solve the problem*

$$\min_{\text{causal } K(\cdot)} \left\| \begin{bmatrix} I - K(z)H(z) & -K(z) \end{bmatrix} \right\|_\infty \stackrel{\Delta}{=} \gamma_{opt}.$$

(i) Suppose $p > m$. Then if $H(z)$ has no zeros on or outside the unit circle

$$\gamma_{opt} < 1. \tag{80}$$

Otherwise $\gamma_{opt} = 1$.

(ii) Suppose $p < m$. *Then*

$$\gamma_{opt} = 1. \tag{81}$$

Proof: To prove part (i) let us first assume that $H(z)$ has no zeros on $|z| \geq 1$. Since $p > m$, This means that $H(z)$ has a causal left inverse, *i.e.*, there exists a $m \times p$ transfer matrix $H^{-L}(z)$, analytic on $|z| \geq 1$, such that

$$H^{-L}(z)H(z) = I_m.$$

Since $H^{-L}(z)$ is analytic on $|z| \geq 1$, for any $\lambda > \|H^{-L}(z)\|_\infty^2$, we can write

$$H^{-L}(e^{j\omega})H^{-L*}(e^{j\omega}) < \lambda I, \quad \forall \omega.$$

Now let us choose our equalizer as $K(z) = \frac{1}{1+\lambda} \cdot H^{-L}(z)$. For this choice we have

$$T_K(z) = \left[\begin{array}{cc} \frac{\lambda}{1+\lambda} \cdot I & -\frac{1}{1+\lambda} \cdot H^{-L}(z) \end{array} \right].$$

Thus

$$\begin{array}{rcl} T_K(e^{j\omega})T_K^*(e^{j\omega}) & = & \dfrac{\lambda^2}{(1+\lambda)^2}I + \dfrac{1}{(1+\lambda)^2}H^{-L}(e^{j\omega})H^{-L*}(e^{j\omega}) \\ & < & \dfrac{\lambda^2}{(1+\lambda)^2}I + \dfrac{\lambda}{(1+\lambda)^2}I = \dfrac{\lambda}{1+\lambda}I < I, \end{array}$$

so that $\|T_K(z)\|_\infty < 1$.

Let us now assume that $H(z)$ has zeros on $|z| \geq 1$. We shall assume that there exists a causal $K(z)$ such that $\|T_K(z)\|_\infty < 1$ and shall obtain a contradiction. Thus assume that there exists a $K(z)$, analytic on $|z| \geq 1$, such that

$$\left(I - K(e^{j\omega})H(e^{j\omega})\right)\left(I - K(e^{j\omega})H(e^{j\omega})\right)^* + K(e^{j\omega})K^*(e^{j\omega}) < I, \quad \forall \omega.$$

Since $T_K(z)$ is analytic on $|z| \geq 1$, the maximum-modulus principle implies that

$$\left(I - K(z)H(z)\right)\left(I - K(z)H(z)\right)^* + K(z)K^*(z) < I, \quad \forall |z| \geq 1. \tag{82}$$

But assume that $H(z)$ has a zero in $|z| \geq 1$, *i.e.*, $H(w)$ has rank less than m for some w such that $|w| \geq 1$. Let X be a corresponding right null vector of $H(w)$, *i.e.*, $H(w)X = 0$. Then we have $[I - K(w)H(w)]X = X$, so that unity is an eigenvalue of $I - K(w)H(w)$, and hence

$$\bar{\sigma}\left[I - K(w)H(w)\right] \geq 1,$$

which clearly contradicts (82).

Finally, to prove (ii) we note that when $p < m$, the number of signals to estimated is greater than the number of obsercations, and so there is no hope in estimating them. $K(z) = 0$ clearly gives $\gamma_{opt} = 1$.

$\blacksquare$

Remark: Note that when $p > m$, $H(z)$ will generically have no zeros because it will generically have full rank for all z. To be more explicit, suppose that $p = 2$ and $m = 1$, so that

$$H(z) = \begin{bmatrix} H_1(z) \\ H_2(z) \end{bmatrix}. \tag{83}$$

Now $H(z)$ will have a zero outside the unit circle if, and only if, $H_1(z)$ and $H_2(z)$ *share* some non-minimum phase zero. But of course any two arbitrary rational functions will generically not have common zeros.

We should also mention that, when $H(z)$ has no zeros in $|z| \geq 1$, finding an explicit expression for γ_{opt} does not appear to be simple. The construction presented in the proof of Theorem 5 suggests the following upper bound

$$\gamma_{opt}^2 \leq \inf_{H^{-L}(\cdot), H^{-L}(z)H(z)=I_m} \frac{\|H^{-L}(z)\|_\infty^2}{1 + \|H^{-L}(z)\|_\infty^2}. \tag{84}$$

Various other statements can also be made. Here is an example. Suppose that $H(z) = \begin{bmatrix} H_1(z) \\ H_2(z) \end{bmatrix}$ where both $H_1(z)$ and $H_2(z)$ are scalar and minimum phase. Then the best causal equalizer performs better than the best noncausal equalizers applied to $H_1(z)$ and $H_2(z)$ separately, but not as good as the best noncausal equalizer applied to $H_1(z)$ and $H_2(z)$ together. In other words,

$$\sup_{\omega \in [-\pi, \pi]} \frac{1}{1 + |H_1(e^{j\omega})|^2 + |H_2(e^{j\omega})|^2} \leq \gamma_{opt}^2 \leq \min_{i=1,2} \sup_{\omega \in [-\pi, \pi]} \frac{1}{1 + |H_i(e^{j\omega})|^2}. \tag{85}$$

6 Some Remedies for Non-Minimum Phase Channels

In Sec. 5.1.4 we noted that, in the square case ($p = m$), causal equalization of non-minimum phase channels is not possible (from the H^∞ point of view). Since non-minimum phase channels do occur in practice it is important to have a means of circumventing this drawback. In this section we shall look at three remedies for the non-minimum phase case, namely, using multiple sensors, oversampling, or allowing for finite delay, as suggested by the framework of this paper.

6.1 Using Multiple Sensors

By adding multiple sensors we increase the number of observations signals without changing the number of input signals. Thus, we transform a square problem with $p = m$ to a non-square problem with $p > m$. This means that our original problem which had $\gamma_{opt} = 1$, when $H(z)$ is non-minimum phase, is transformed to one for which γ_{opt} is generically strictly less than one.

Take for example the scalar case and suppose that $H(z)$ is non-minimum phase. Then if we add a second sensor, corresponding to a scalar channel with transfer function $H'(z)$, our channel will become

$$\begin{bmatrix} H(z) \\ H'(z) \end{bmatrix}.$$

Thus, even though $H(z)$ is non-minimum phase, and $H'(z)$, too, may very well be non-minimum phase, it is still possible to causally equalize the aggregate channel as long as $H(z)$ and $H'(z)$ share no non-minimum phase zeros. This, of course, generically holds.

We should also mention that the property of non-common zeros is similar to the requirement that arises in the blind equalization of vector channels using second-order statistics (see *e.g.*, [22]).

6.2 Oversampling

Oversampling is also currently used as a method for the blind equalization of non-minimum phase channels using second-order statistics (see *e.g.* [23]), and can also be used for causal equalization. The idea is to sample the output of the communications channel at a rate higher than the rate of the input signal. This then leads us to a situation where we have more observations signals than input signals, which can be suitably transformed to a multi-input multi-output (MIMO) channel scenario with $p > m$ that was considered in Sec.6.1. Indeed if we have a non-minimum phase scalar channel with transfer function $H(z)$, and oversample at a rate of r output samples per each input transmitted, the resulting MIMO channel becomes

$$\begin{bmatrix} H(z) \\ H(e^{j\frac{2\pi}{r}}z) \\ \vdots \\ H(e^{j\frac{2(r-1)\pi}{r}}z) \end{bmatrix}. \tag{86}$$

Thus the condition for causal H^∞ equalization is that the transfer functions $\{H(e^{j\frac{2k\pi}{r}}z)\}_{k=0}^{r-1}$ share no common non-minimum phase zero.

6.3 Finite Delay

Another obvious remedy for the equalization of non-minimum phase channels is to allow for some finite delay, $d > 0$. In other words, we are interested in estimating u_i using the observations $\{\ldots y_{i+d-1}, y_{i+d}\}$. Such a problem corresponds to the general estimation problem of Sec. 3 with $L(z) = z^{-d}$ (since we can also think of it as estimating u_{i-d} using the observations $\{\ldots y_{i-1}, y_i\}$).

It is quite clear that allowing for infinite delay takes us back to the situation of a noncausal equalizer. The natural question to ask therefore is: what is the amount of finite delay, $0 < d < \infty$, that we should incur so that

(i) $\gamma_{opt} < 1$?

(ii) $\gamma_{opt} = \gamma_s$?

The answer to the first question would tell how much delay we need in order to be successful in causal equalization, and the answer to the second question would tell us how much delay we need to in order obtain the same performance as the non-causal equalizer.

In order to illustrate the effect of delay, and to answer the above questions, it will be instructive to look at the case of a scalar single-zero FIR channel,

$$H(z) = 1 + az^{-1}.$$

Lemma 6 (Equalization of Single-Zero Channel) *Consider the scalar single-zero FIR channel $H(z) = 1 + az^{-1}$ and suppose we want to solve the problem*

$$\min_{\text{causal} K(\cdot)} \left\| \begin{bmatrix} z^{-d} - K(z)H(z) & -K(z) \end{bmatrix} \right\|_\infty \triangleq \gamma_{opt},$$

for $d \geq 0$. Then we have the following result.

(i) If $d = 0$, then

$$\gamma_{opt}^2 = \begin{cases} \max_{\omega\in[0,2\pi]} \frac{1}{1+|H(e^{j\omega})|^2} & \text{if } |a| < 1 \\ 1 & \text{if } |a| \geq 1 \end{cases}. \tag{87}$$

(ii) If $d = 1$, then

$$\gamma_{opt}^2 = \begin{cases} \max_{\omega\in[0,2\pi]} \frac{1}{1+|H(e^{j\omega})|^2} & \text{if } |a| < 2 \\ \frac{2}{|a|^2} & \text{if } |a| \geq 2 \end{cases}. \tag{88}$$

(iii) If $d \geq 2$, then

$$\gamma_{opt}^2 = \max_{\omega\in[0,2\pi]} \frac{1}{1+|H(e^{j\omega})|^2}. \tag{89}$$

Proof: The proof for part (i) follows readily from Theorem 4, since $H(z)$ is minimum phase for $|a| < 1$ and non-minimum phase otherwise.

To prove part (ii), we note that in $d = 1$ case, the transfer matrix whose J-spectral factorization we must compute is given by

$$\begin{bmatrix} 1 + H(z)H^*(z^{-*}) & -H(z)L^*(z^{-*}) \\ -L(z)H^*(z^{-*}) & L(z)L^*(z^{-*}) - \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 + (1 + az^{-1})(1 + a^*z) & -(1 + az^{-1})z \\ -z^{-1}(1 + a^*z) & 1 - \gamma^2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 + |a|^2 + az^{-1} + a^*z & -(z + a) \\ -(z^{-1} + a^*) & (1 - \gamma^2) \end{bmatrix}.$$

We first observe that, unlike the transfer matrix for the $d = 0$ case of Eq. (38), the $(1, 2)$ component is anti-causal and the $(2, 1)$ component is causal. Therefore the UDL factorization used there will not lead us to the desired J-spectral factorization.

However, if we consider the following factorization

$$\begin{bmatrix} 2 + |a|^2 + az^{-1} + a^*z & -(z + a) \\ -(z^{-1} + a^*) & (1 - \gamma^2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{a} & 1 \end{bmatrix} \begin{bmatrix} 2 + |a|^2 + az^{-1} + a^*z & \frac{2}{a^*} + \frac{a}{a^*}z^{-1} \\ \frac{2}{a} + \frac{a^*}{a}z & \frac{2}{|a|^2} - \gamma^2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{a^*} \\ 0 & 1 \end{bmatrix}, \tag{90}$$

we observe that the center matrix takes the form of the $d = 0$ case, since the $(1, 2)$ component is now causal and the $(2, 1)$ component is now anti-causal. In fact, by making the following substitutions

$$H'(z) = -(a + \frac{a^2}{2}z^{-1}) \ , \ L'(z) = 1 \ , \ r' = \frac{|a|^2}{2} \ , \ q' = \frac{2}{|a|^2}$$

the desired transfer matrix can be written as

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{a} & 1 \end{bmatrix} \begin{bmatrix} r' + q'H'(z)H'^*(z^{-*}) & -q'H'(z)L'^*(z^{-*}) \\ -q'L'(z)H'^*(z^{-*}) & q'L'(z)L'^*(z^{-*}) - \gamma^2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{a} \\ 0 & 1 \end{bmatrix}.$$

From the $d = 0$ case, we already have a factorization for the center matrix:

$$\begin{bmatrix} r' + q'H'(z)H'^*(z^{-*}) & -q'H'(z)L'^*(z^{-*}) \\ -q'L'(z)H'^*(z^{-*}) & q'L'(z)L'^*(z^{-*}) - \gamma^2 \end{bmatrix} = \begin{bmatrix} L'_{11}(z) & L'_{12}(z) \\ L'_{21}(z) & L'_{22}(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} L'^*_{11}(z^{-*}) & L'^*_{21}(z^{-*}) \\ L'^*_{12}(z^{-*}) & L'^*_{22}(z^{-*}) \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{a} & 1 \end{bmatrix} \begin{bmatrix} L'_{11}(z) & L'_{12}(z) \\ L'_{21}(z) & L'_{22}(z) \end{bmatrix} = \begin{bmatrix} L'_{11}(z) & L'_{12}(z) \\ L'_{21}(z) - \frac{L'_{11}(z)}{a} & L'_{22}(z) - \frac{L'_{12}(z)}{a} \end{bmatrix},$$

which is the desired J-spectral factor since, all entries are causal, $L_{11}(z)$, which is equal to $L'_{11}(z)$, is minimum phase, $L_{12}(z)$, which is equal to $L'_{12}(z)$ is strictly causal and the whole matrix is causally invertible since it is the product of two causal and causally invertible matrices.

We thus need to distinguish between the two cases where $H'(z)$ is minimum phase and where $H'(z)$ is non-minimum phase. When $H'(z)$ is minimum phase, which is the case for $|a| < 2$, then

$$\gamma_{opt}^2 = \sup_{\omega\in[0,2\pi]} \frac{r'q'}{r' + q'|H'(e^{j\omega})|^2} = \sup_{\omega\in[0,2\pi]} \frac{1}{\frac{a^2}{2} + \frac{2}{a^2}|a + \frac{a^2}{2}e^{-j\omega}|^2}.$$

After some simplification, the expression for γ_{opt}^2 becomes

$$\gamma_{opt}^2 = \sup_{\omega\in[0,2\pi]} \frac{1}{1 + |1 + ae^{-j\omega}|^2} = \sup_{\omega\in[0,2\pi]} \frac{1}{1 + |H(e^{j\omega})|^2} = \gamma_s^2.$$

When $H'(z)$ is non-minimum phase, which is the case for $|a| \geq 2$, then

$$\gamma_{opt}^2 = q' = \frac{2}{a^2} \ .$$

To prove part (iii), we note that in the $d = 2$ case, the transfer matrix whose J-spectral factorization we must compute is given by

$$\begin{bmatrix} 2 + |a|^2 + az^{-1} + a^*z & -(z^2 + az) \\ -(z^{-2} + a^*z^{-1}) & (1 - \gamma^2) \end{bmatrix}.$$

By a trick similar to the one used in the $d = 1$ case, we can consider the following factorization

$$\begin{aligned} &\begin{bmatrix} 2 + |a|^2 + az^{-1} + a^*z & -(z + a) \\ -(z^{-1} + a^*) & (1 - \gamma^2) \end{bmatrix} = \\ &\begin{bmatrix} 1 & 0 \\ -\frac{z^{-1}}{a} + \frac{2}{a^2} & 1 \end{bmatrix} \begin{bmatrix} 2 + |a|^2 + az^{-1} + a^*z & -(\frac{4+|a|^2}{a^2} + \frac{2a^*}{a^2}z^{-1}) \\ -(\frac{4+|a|^2}{a^{*2}} + \frac{2a}{a^{*2}}z) & \frac{8+2|a|^2}{|a|^4} - \gamma^2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{z}{a^*} + \frac{2}{a^{*2}} \\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{91}$$

and make the following substitutions

$$H'(z) = (\frac{a^{*2}}{2} + \frac{a^{*3}}{4 + |a|^2}z^{-1}) \ , \ L'(z) = 1 \ , \ r' = \frac{|a|^4}{8 + 2|a|^2} \ , \ q' = \frac{8 + 2|a|^2}{|a|^4}.$$

Then it is readily seen that Eq. (91) can be rewritten as

$$\begin{aligned} &\begin{bmatrix} 2 + |a|^2 + az^{-1} + a^*z & -(z + a) \\ -(z^{-1} + a^*) & (1 - \gamma^2) \end{bmatrix} = \\ &\begin{bmatrix} 1 & 0 \\ -\frac{z^{-1}}{a} + \frac{2}{a^2} & 1 \end{bmatrix} \begin{bmatrix} r' + q'H'(z)H'^*(z^{-*}) & -q'H'(z) \\ -q'H'^*(z^{-*}) & q' - \gamma^2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{z}{a^*} + \frac{2}{a^{*2}} \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{92}$$

Once again we can factorize the center matrix using results from the $d = 0$ case to obtain the desired J-spectral factorization. An important result here is that $H'(z)$ is minimum phase for all values of a, since its zero lies at $-\frac{2a^*}{4+|a|^2}$, and

$$\left| \frac{2a^*}{4 + |a|^2} \right| \leq 1.$$

This means that the min-max energy gain is given by

$$\gamma_{opt}^2 = \frac{r'q'}{r' + q' \min_w |H'(e^{jw})|^2}.$$

Once more, algebraic simplifications show that

$$\gamma_{opt}^2 = \sup_{\omega \in [0,2\pi]} \frac{1}{1 + |H(e^{j\omega})|^2} = \gamma_s^2.$$

∎

Lemma 6 shows that, for the channel $H(z) = 1 + az^{-1}$, allowing a single delay in equalization, results in extending the region for which $\gamma_{opt} = \gamma_s$ from inside the unit circle to inside a circle of radius two. Moreover, there is no region for which $\gamma_{opt} = 1$, which means that a delay of one unit is sufficient to perform causal equalization for all single-zero FIR channels. Finally, for $d \geq 2$, we have $\gamma_{opt}^2 = \gamma_s^2$, *i.e.*, a delay of two units is sufficient to obtain the same H^∞ performance as the noncausal solution in equalizing a single-zero FIR channel.

Unfortunately, we do not yet know an explicit factorization for the transfer matrix

$$\begin{bmatrix} I + H(z)H^*(z^{-*}) & -z^d H(z) \\ -z^{-d} H^*(z^{-*}) & (1-\gamma^2)I \end{bmatrix}, \tag{93}$$

for arbitrary $d > 0$ and general non-minimum phase channels. Therefore we do not yet have explicit expressions for γ_{opt} in this case and have no answer for question (ii), raised above, of when $\gamma_{opt} = \gamma_s$. However, we do have the answer to question (i), of when $\gamma_{opt} = 1$.

Lemma 7 (Delay Required for $\gamma_{opt} < 1$) *Consider the $m \times m$ causal rational transfer matrix $H(z) = h_0 + h_1 z^{-1} + \ldots$ and suppose that $H(z)$ has no unit circle zeros (*i.e., *$H(e^{j\omega}$ is full rank for all ω). We are interested in the following problem:*

$$\min_{\text{causal} K(z)} \left\| \begin{bmatrix} z^{-d}I - K(z)H(z) & -K(z) \end{bmatrix} \right\|_\infty \triangleq \gamma_{opt} \quad , d \geq 0. \tag{94}$$

*Let l denote the number of non-minimum phase (*i.e., *on or outside the unit circle) zeros of $H(z)$ (counting multiplicities). Then we have the following result.*

(i) If $d \geq l$, then

$$\gamma_{opt} < 1.$$

(ii) If $d < l$, then

$$\gamma_{opt} = 1.$$

Proof: The proof of this lemma is given in [24].

■

The above lemma has an interesting interpretation. In order to get an improvement over not equalizing at all ($\gamma_{opt} = 1$), the delay d should be chosen greater than the number of non-minimum phase zeros of the channel. Thus the fundamental limitation in causal equalization is the number of zeros outside of the unit circle.

7 State-Space Formulae

The key to our results in this paper is the explicit J-spectral factorization of the transfer matrix (38). We performed this factorization in the input-output setting of transfer matrices, and in the square case of $p = m$. We should mention that most results in H^∞ theory are in state-space language, and it is therefore possible to give expressions for this factorization using state-space techniques. This is especially useful in the non-square case ($p \neq m$) where we do not have explicit expressions for the factorization. Moreover, state-space tools provide a powerful and efficient means for actual computation of the spectral factors. Nonetheless, the solutions are not as explicit, and we believe that the results of this paper would have been much more difficult to conceive in the state-space setting.

In this section we shall give a very brief outline of the state-space solutions. These are given in terms of solutions of so-called discrete-time algebaric Riccati equations (DARE's) that allow one to perform the required J-spectral factorization. The relationship between algebaric Riccati equations and solutions to H^∞ problems is well known and can be found in any textbook on H^∞ theory [6, 25, 26].

In the state-space framework the channel $H(z)$ is assumed to have a finite-dimensional linear time-invariant state-space model of the form

$$\left\{ \begin{array}{rcl} x_{i+1} & = & Ax_i + Bu_i \\ o_i & = & Cx_i + Du_i \end{array} \right. , \tag{95}$$

where $A \in \mathcal{C}^{n\times n}$, $B \in \mathcal{C}^{n\times m}$, $C \in \mathcal{C}^{p\times n}$ and $D \in \mathcal{C}^{p\times m}$. Note that we may write $H(z) = D + C(zI - A)^{-1}B$. In particular, when $H(z)$ is FIR,

$$H(z) = h_0 + h_1 z^{-1} + \ldots h_l z^{-l}, \tag{96}$$

as is often the case, one possible state-space description is given by

$$A = \begin{bmatrix} 0 & & & & \\ I & 0 & & & \\ & I & \ddots & & \\ & & \ddots & \ddots & \\ & & & I & 0 \end{bmatrix}, \ B = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} h_1 & h_2 & \dots & h_l \end{bmatrix}, \ D = h_0. \tag{97}$$

Then we have the following result.

Theorem 6 (State-Space Formulae) *Consider the $p \times m$ rational transfer matrix $H(z) = D + C(zI - A)^{-1}B$, with $\{A, B, C\}$ minimal. Then a causal $K(z)$ that satisfies*

$$\left\| \begin{bmatrix} I_m - K(z)H(z) & -K(z) \end{bmatrix} \right\|_\infty < \gamma,$$

exists if, and only if, there exists a solution to the DARE

$$P = FPF^* + GG^* - K_p R_e K_p^*, \tag{98}$$

with

$$K_p = \begin{bmatrix} APC^* + GD^* & G \end{bmatrix} R_e^{-1} \quad , \quad R_e = \begin{bmatrix} I_p + DD^* & D \\ D^* & (1-\gamma^2)I_m \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} P \begin{bmatrix} C^* & 0 \end{bmatrix} \tag{99}$$

such that

(i) $A_p \triangleq A - K_p \begin{bmatrix} C \\ 0 \end{bmatrix}$ *is stable.*

(ii) $P \geq 0$.

(iii) R_e *and* $\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}$ *have the same inertia*[4].

If this is the case, then the $L_{ij}(z)$ in the canonical factorization of (38) are given by

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \left(\begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} (zI - A)^{-1} K_p \right) R_e^{1/2}, \tag{100}$$

[4]By the inertia of a Hermitian matrix, we mean the number of its positive, negative and zero eigenvalues. A simple way of calculating the inertia of a strongly regular Hermitian matrix R (*i.e.* one whose leading minors are all nonzero), is by computing its LDU decomposition

$$R = LDL^*,$$

where L is a lower triangular matrix with unit diagonal and D is a diagonal matrix: the number of positive and negative elements of D give the number of positive and negative eigenvalues of R and hence the inertia.

where $R_e^{1/2}$ *is found from the block LDU factorization* $R_e = R_e^{1/2}\begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} R_e^{*/2}$. *All solutions* $K(z)$ *are given by*

$$K(z) = \left(L_{22}(z)S(z) - L_{21}(z)\right)\left(L_{11}(z) - L_{12}(z)S(z)\right)^{-1},$$

where $S(z)$ *is any causal and strictly contractive transfer matrix. In particular, defining* $R_{He} \triangleq I_p + DD^* + CPC^*$, *we have*

$$\left\{ \begin{array}{lcl} L_{11}(z) & = & \left[I_p + C(zI-A)^{-1}(APC^* + GD^*)R_{He}^{-1}\right] R_{He}^{1/2} \\ L_{21}(z) & = & D^* R_{He}^{-*/2} \end{array} \right. , \tag{101}$$

so that

$$K_{cen}(z) = D^* R_{He}^{-1} + D^* R_{He}^{-1} C(zI - A_1)^{-1}(APC^* + GD^*)R_{He}^{-1}, \tag{102}$$

where $A_1 \triangleq A - (APC^* + GD^*)R_{He}^{-1}C$.

Proof: The proof can be found in [6, 26].

■

Finally, we may remark that Theorem 4, in conjunction with the above result, allow us to make the following statement about the DARE (98).

Lemma 8 (Properties of the DARE) *Consider the* $m \times m$ *rational transfer matrix* $H(z) = D + C(zI - A)^{-1}B$, *with* $\{A, B, C\}$ *minimal. Then a solution to the DARE (98) with the properties (i)-(iii) of Theorem 6 exists if, and only if,*

(i)

$$\gamma > \left\| \left(I_m + H^*(z^{-*})H(z)\right)^{-1} \right\|_\infty^{1/2},$$

if $A - BD^{-1}C$ *is stable (has all its eigenvalues strictly inside the unit circle).*

(i)

$$\gamma > 1,$$

if $A - BD^{-1}C$ *is unstable (has eigenvalues on and outside the unit circle).*

Proof: The proof follows from Theorems 4 and 6 and the fact that

$$H^{-1}(z) = D^{-1} - D^{-1}C(zI - A + BD^{-1}C)^{-1}BD^{-1},$$

so that $H(z)$ is minimum phase if, and only if, $A - BD^{-1}C$ is stable.

■

We should remark that a state-space proof of our main result (Theorem 4) would have required us to give a direct proof of Lemma 8, which appears to be highly nontrivial.

8 Conclusion

In this paper, we studied the H^∞ criterion as an alternative method for the equalization of communication channels. Current equalization methods and algorithms are mainly concerned with average behavior (*e.g.*, MSE or BER) under the conditions of perfect models and known statistical distributions. H^∞ theory, on the other hand, is concerned with the worst-case behavior and the question of robustness with respect to model uncertainty and lack of statistical information.

We obtained a closed-form solution to the H^∞ equalization problem and parametrized all possible H^∞-optimal equalizers. More importantly, not only did we obtain a solution to the problem, but we learned a lot about the equalization problem itself. In particular, we discovered an interesting dichotomy between minimum phase and non-minimum phase channels: In the minimum phase case causal equalizers perform as well as non-causal ones, whereas in the non-minimum phase case, causal equalizers cannot reduce the worst-case estimation error bounds from their a priori values. Moreover, we showed that, for minimum phase channels, a scaled version of the zero-forcing equalizer is H^∞-optimal. We also studied the use of adding more sensors, oversampling, and allowing for finite delay, as remedies for the non-minimum phase case. We also learned that the fundamental limit in causally equalizing a non-minimum phase channel is its number of non-minimum phase zeros.

The results presented in this paper also have ramifications in other areas. They essentially deal with the problem of "robustly" inverting a linear time-invariant system and so have implications to the important tracking problem of control theory [27]. They also have various implications to more general control and estimation problems and, especially, to the questions of worst-case controllability and estimability (essentially what systems are easy, or difficult, to control and estimate) [24].

Finally, we may mention some possible directions for future work. The first has to do with the use of finite delay, and especially the question of how much delay is required to obtain the same H^∞ performance as the noncausal solution? Another has to do with incorporating "structural" modeling errors for the channel in the equalizer solution [21]. Perhaps the most important direction relies on the following observation: As we saw for H^∞-optimal zero-forcing equalization, having a good (indeed optimal) worst-case performance alone is not sufficient for having acceptable performance in real applications.

Since the H^∞ criterion provides a family of filters which achieve the same worst-case performance, it is an important design problem to choose a filter among this family which has good performance with respect to other relevant criteria.

A Proof of Theorem 2

First note that the expression for $T_K(z)$ allows us to write

$$\gamma^2 I - T_K(z)T_K^*(z^{-*}) = \begin{bmatrix} K(z) & I \end{bmatrix} \begin{bmatrix} I + H(z)H^*(z^{-*}) & -H(z)L^*(z^{-*}) \\ -L(z)H^*(z^{-*}) & -\gamma^2 I + L(z)L^*(z^{-*}) \end{bmatrix} \begin{bmatrix} K^*(z^{-*}) \\ I \end{bmatrix}. \tag{A.1}$$

We will first prove sufficiency. Thus suppose that the factorization (19) can be performed. Then replacing the center matrix of (A.1) with its factored form from (19), we readily see that we may write

$$\begin{aligned} \gamma^2 I - T_K(z)T_K^*(z^{-*}) &= (K(z)L_{11}(z) + L_{21}(z)) \left(K(z^{-*})L_{11}(z^{-*}) + L_{21}(z^{-*})\right)^* \\ &\quad - (K(z)L_{12}(z) + L_{22}(z)) \left(K(z^{-*})L_{12}(z^{-*}) + L_{22}(z^{-*})\right)^*. \end{aligned} \tag{A.2}$$

Since $L_{11}^{-1}(z)$ and $L_{21}(z)$ are causal, $K_{cen}(z) = -L_{21}(z)L_{11}^{-1}(z)$ is a causal filter. Moreover, it sets the first term on the RHS of (A.2) to zero and yields

$$\gamma^2 I - T_{K_{cen}}(z)T_{K_{cen}}^*(z^{-*}) = -\left(L_{22}(z) - L_{21}(z)L_{11}^{-1}(z)L_{12}(z)\right)\left(L_{22}(z^{-*}) - L_{21}(z^{-*})L_{11}^{-1}(z^{-*})L_{12}(z^{-*})\right)^*, \tag{A.3}$$

which is clearly negative on the unit circle. Thus $\|T_{K_{cen}}(z)\|_\infty < \gamma$ and the suboptimal H^∞ problem has a solution.

Note, moreover, from (A.2) that all $K(z)$ that yield $\gamma^2 I - T_K(z)T_K^*(z^{-*})$ on the unit circle negative, are given by

$$(K(z)L_{11}(z) + L_{21}(z)) = (K(z)L_{12}(z) + L_{22}(z))\, S(z), \tag{A.4}$$

where $S(z)$ is any causal and strictly contractive transfer matrix. Solving for $K(z)$ in (A.4), yields (20). What, of course, remains to be shown is that for any causal and strictly contractive $S(z)$ the transfer matrix $K(z)$ is causal.[5] To show this, let us write

$$K(z) = (L_{22}(z)S(z) - L_{21}(z))\left(I - L_{11}^{-1}(z)L_{12}(z)S(z)\right)^{-1} L_{11}^{-1}(z), \tag{A.5}$$

[5] We could lose the causality of $K(z)$ if $(L_{11}(z) - L_{12}(z)S(z))^{-1}$ is not causal.

and note that the causal transfer matrix $A(z) \stackrel{\Delta}{=} L_{11}^{-1}(z)L_{12}(z)S(z)$ is strictly contractive since,

$$\begin{aligned}
A(e^{j\omega})A(e^{j\omega})^* &= L_{11}^{-1}(e^{j\omega})L_{12}(e^{j\omega})S(e^{j\omega})S^*(e^{j\omega})L_{12}^*(e^{j\omega})L_{11}^{-*}(e^{j\omega}) \\
&< L_{11}^{-1}(e^{j\omega})L_{12}(e^{j\omega})L_{12}^*(e^{j\omega})L_{11}^{-*}(e^{j\omega}), \\
&= L_{11}^{-1}(e^{j\omega})(L_{11}(e^{j\omega})L_{11}^*(e^{j\omega}) - I - H(e^{j\omega})H^*(e^{j\omega}))L_{11}^{-*}(e^{j\omega}), \\
&< L_{11}^{-1}(e^{j\omega})L_{11}(e^{j\omega})L_{11}^*(e^{j\omega})L_{11}^{-*}(e^{j\omega}) \\
&= I,
\end{aligned}$$

where in the second step we used $S(e^{j\omega})S^*(e^{j\omega}) < I$, and in the third step we equated the $(1,1)$ entries in (19). Appealing to Lemma 9 of App. B, this means that $\left(I - L_{11}^{-1}(z)L_{12}(z)S(z)\right)^{-1}$ is causal, and hence so is $K(z)$ since it is the product of causal transfer matrices.

The proof of necessity is slightly more detailed. The proof presented here is of independent interest since it also establishes the maximum entropy property of the central solution, (21). To this end, suppose that a causal $K(z)$ exists that solves the H^∞ estimation problem of level γ. In other words, there exists a causal $K(z)$ such that

$$\gamma^2 I - T_K(e^{j\omega})T_K^*(e^{j\omega}) > 0, \quad \forall \omega \in [0, 2\pi]. \tag{A.6}$$

Therefore we may define the entropy cost

$$\epsilon_K = \int_0^{2\pi} \log \det \left[\gamma^2 I - T_K(e^{j\omega})T_K^*(e^{j\omega})\right] d\omega. \tag{A.7}$$

Note that, over the set (A.6), the entropy cost is a concave function of $K(\cdot)$ and that its value approaches negative infinity as $K(\cdot)$ approaches the boundary of (A.6). Therefore since the set defined via (A.6) is concave and nonempty (we have assumed that a causal H^∞ estimator of level γ exists), there will exist a unique causal transfer matrix, which we denote by $K_{cen}(\cdot)$, that maximizes the entropy subject to the constraint (A.6). In other words, there exists a solution to the following convex optimization problem,

$$\left\{ \begin{array}{ll} \max_{\text{causal } K(\cdot)} & \epsilon_K \\ & \\ \text{subject to} & \text{(A.6)} \end{array} \right. . \tag{A.8}$$

Now since $K_{cen}(\cdot)$ lies inside the constraint set, it is straightforward to see that it should make the derivative of the entropy cost with respect to causal $K(\cdot)$ equal to zero. Computing this derivative with respect to $\{k_i, i \geq 0\}$, the impulse response coefficients of $K(z) = k_0 + k_1 z^{-1} + \ldots$, yields

$$\frac{\partial \epsilon_K}{\partial k_i} = \int_0^{2\pi} \frac{\partial \log \det \left[\gamma^2 I - T_K(e^{j\omega})T_K^*(e^{j\omega})\right]}{\partial [T_K(e^{j\omega})T_K^*(e^{j\omega})]} \cdot \frac{\partial [T_K(e^{j\omega})T_K^*(e^{j\omega})]}{\partial T_K(e^{j\omega})} \cdot \frac{\partial T_K(e^{j\omega})}{\partial K(e^{j\omega})} \cdot \frac{\partial K(e^{j\omega})}{\partial k_i} d\omega$$

$$= \int_0^{2\pi} \left[\gamma^2 I - T_K(e^{j\omega})T_K^*(e^{j\omega})\right]^{-1} \cdot 2T_K(e^{j\omega}) \cdot \begin{bmatrix} -H^*(e^{j\omega}) \\ -I \end{bmatrix} \cdot e^{-j\omega i} d\omega$$

$$= 2\int_0^{2\pi} \left[\gamma^2 I - T_K(e^{j\omega})T_K^*(e^{j\omega})\right]^{-1} \left[K(e^{j\omega})\left(I + H(e^{j\omega})H^*(e^{j\omega})\right) - L(e^{j\omega})H^*(e^{j\omega})\right] e^{-j\omega i} d\omega.$$

Note that the RHS of the last equation is simply the coefficient of z^{-i} in the Laurent expansion of the transfer matrix $2\left[\gamma^2 I - T_K(z)T_K^*(z^{-*})\right]^{-1}\left[K(z)\left(I + H(z)H^*(z^{-*})\right) - L(z)H^*(z^{-*})\right]$. Since we require that $\frac{\partial \epsilon_K}{\partial k_i} = 0$ for all $i \geq 0$, this implies that the causal coefficients of this transfer matrix must be zero. Thus $K_{cen}(\cdot)$ satisfies the following functional equation

$$\left\{\left[\gamma^2 I - T_{K_{cen}}(z)T_{K_{cen}}^*(z^{-*})\right]^{-1}\left[K_{cen}(z)\left(I + H(z)H^*(z^{-*})\right) - L(z)H^*(z^{-*})\right]\right\}_+ = 0, \tag{A.9}$$

where the notation $\{A(z)\}_+$ denotes the causal (analytic on $|z| \geq 1$) part of $A(z)$.

To summarize what has been shown so far, if the H^∞ estimation problem has a solution then (A.9) has a causal solution that satisfies the constraint (A.6).

To solve (A.9) let us first introduce the canonical spectral factorization

$$L_\Delta(z)L_\Delta^*(z^{-*}) = \gamma^2 I - T_{K_{cen}}(z)T_{K_{cen}}^*(z^{-*}), \tag{A.10}$$

with $L_\Delta(z)$ causal and causally invertible. In view of (A.6), this factorization can always be performed. With this definition (A.9) can be written as

$$\left\{L_\Delta^{-*}(z^{-*})L_\Delta^{-1}(z)\left[L(z)H^*(z^{-*}) - K_{cen}(z)\left(I + H(z)H^*(z^{-*})\right)\right]\right\}_+ = 0. \tag{A.11}$$

The above equation implies that the transfer matrix appearing inside the $\{\cdot\}_+$ must be strictly anticausal. Denoting this strictly anticausal transfer matrix by $B(z)$, we readily see that, since $L_\Delta^{-*}(z^{-*})$ is anticausal, $L_\Delta^*(z^{-*})B(z)$ is also strictly anticausal. Thus the transfer matrix

$$L_{12}^*(z^{-*}) \stackrel{\Delta}{=} L_\Delta^{-1}(z)\left[L(z)H^*(z^{-*}) - K_{cen}(z)\left(I + H(z)H^*(z^{-*})\right)\right], \tag{A.12}$$

is strictly anticausal. Moreover, let us define the causal and causally invertible, transfer matrix $L_{11}(z)$ via the canonical factorization

$$L_{11}(z)L_{11}^*(z^{-*}) = I + H(z)H^*(z^{-*}) + L_{12}(z)L_{12}^*(z^{-*}). \tag{A.13}$$

Finally, for reasons to become clear shortly, let us define the causal operators $L_{21}(z)$ and $L_{22}(z)$ via

$$L_{21}(z) = -K_{cen}(z)L_{11}(z) \quad \text{and} \quad L_{22}(z) = L_\Delta(z) + L_{21}(z)L_{11}^{-1}(z)L_{12}(z). \tag{A.14}$$

Now it is not difficult to see (where for simplicity we have suppressed the dependencies on z),

$$\begin{aligned} L_{21}L_{11}^* - L_{22}L_{12}^* &= -K_{cen}L_{11}L_{11}^* - (L_\Delta + L_{21}L_{11}^{-1}L_{12})L_{12}^* \\ &= -K_{cen}L_{11}L_{11}^* - (L_\Delta - K_{cen}L_{12})L_{12}^* \\ &= -K_{cen}(L_{11}L_{11}^* - L_{12}L_{12}^*) - L_\Delta L_{12}^* \\ &= -K_{cen}(I+HH^*) - LH^* + K_{cen}(I+HH^*) = LH^*, \end{aligned} \tag{A.15}$$

and

$$\begin{aligned} L_{21}L_{21}^* - L_{22}L_{22}^* &= K_{cen}L_{11}L_{11}^*K_{cen}^* - (L_\Delta - K_{cen}L_{12})(L_\Delta - K_{cen}L_{12})^* \\ &= K_{cen}(L_{11}L_{11}^* - L_{12}L_{12}^*)K_{cen}^* + L_\Delta L_{12}^* K_{cen}^* + K_{cen}L_{12}L_\Delta^* - L_\Delta L_\Delta^* \\ &= K_{cen}(I+HH^*)K_{cen}^* + (LH^* - K_{cen}(I+HH^*))K_{cen}^* + \\ &\quad K_{cen}(HL^* - (I+HH^*)K_{cen}^*) + T_{K_{cen}}T_{K_{cen}}^* - \gamma^2 I \\ &= -K_{cen}(I+HH^*)K_{cen}^* + LH^*K_{cen}^* + K_{cen}HL^* + T_{K_{cen}}T_{K_{cen}}^* - \gamma^2 I \\ &= -\gamma^2 I + LL^*. \end{aligned} \tag{A.16}$$

Combining (A.13), (A.15), and (A.16) shows that the causal and causally invertible, $L_{11}(z)$, the strictly causal $L_{12}(z)$, and the causal $L_{21}(z)$ and $L_{22}(z)$ can be found from the factorization (19). Note, moreover, that, from the definition (A.14), we can readily find the central solution as $K_{cen}(z) = -L_{21}(z)L_{11}^{-1}(z)$.

All that remains to show is that $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}^{-1}$ is causal. But this readily follows from the block triangular factorization

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ L_{21}(z)L_{11}^{-1}(z) & I \end{bmatrix} \begin{bmatrix} L_{11}(z) & 0 \\ 0 & \underbrace{L_{22}(z) - L_{21}(z)L_{11}^{-1}(z)L_{12}(z)}_{L_\Delta(z)} \end{bmatrix} \begin{bmatrix} I & L_{11}^{-1}(z)L_{12}(z) \\ 0 & I \end{bmatrix}$$

and the facts that $L_{11}^{-1}(z)$, $L_\Delta^{-1}(z)$ and $L_{21}L_{11}^{-1}$ are all causal.

∎

B Some Important Lemmas

Lemma 9 *Suppose the $n \times n$ transfer matrix $A(z)$ is causal and strictly contractive,* i.e.,

(i) $A(z)$ is analytic on $|z| \geq 1$.

(ii) $A(z)A^*(z^{-*}) < I_n$ *for* $|z| = 1$.

Then the transfer matrix $(I_n - A(z))^{-1}$ *is causal.*

Proof: Since $A(z)$ is a contraction, this implies that $A(e^{j\omega})A^*(e^{j\omega}) < I_n, \forall\omega$. Moreover, since $A(z)$ is analytic on $|z| \geq 1$, due to the maximum modulus principle, it achieves its maximum on the boundary. Thus,

$$A(z)A^*(z) < I_n, \quad \forall |z| \geq 1. \tag{B.17}$$

This means that we can expand the inverse in $(I_n - A(z))^{-1}$ to write

$$(I_n - A(z))^{-1} = I_n + A(z) + A^2(z) + \ldots, \tag{B.18}$$

where the infinite series converges absolutely for $|z| \geq 1$ since $A(z)$ is a strict contraction in this region (see [28]). Thus, $(I_n - A(z))^{-1}$ is the sum of absolutely converging analytic functions on $|z| \geq 1$, and hence itself is an analytic function on $|z| \geq 1$. Thus, $(I_n - A(z))^{-1}$ is causal.

■

Lemma 10 *Suppose that the* $n \times n$ *transfer matrix* $C(z)$ *is a causal strict expansion,* i.e., *that*

(i) $C(z)$ *is analytic on* $|z| \geq 1$.

(ii) $C(z)C^*(z^{-*}) > I_n$ *for* $|z| = 1$.

Then we have

$$C^{-1}(z) \textit{ noncausal} \;\Rightarrow\; (I_n - C(z))^{-1} \textit{ noncausal.}$$

Proof: We shall prove the lemma by contradiction, *i.e.*, we shall assume that $(I_n - C(z))^{-1}$ is causal and prove that $C^{-1}(z)$ is causal. First note that $C(e^{j\omega})C^*(e^{j\omega}) > I_n$ implies

$$\begin{aligned}
& \left(I_n - C(e^{j\omega})\right)^{-1} C(e^{j\omega})C^*(e^{j\omega}) \left(I_n - C(e^{j\omega})\right)^{-*} > \left(I_n - C(e^{j\omega})\right)^{-1} \left(I_n - C(e^{j\omega})\right)^{-*} \\
\Rightarrow\; & \left(I_n - C(e^{j\omega})\right)^{-1} \left[I_n - \left(I_n - C(e^{j\omega})\right)\right] \left[I_n - \left(I_n - C(e^{j\omega})\right)^*\right] \left(I_n - C(e^{j\omega})\right)^{-*} \\
& > \left(I_n - C(e^{j\omega})\right)^{-1} \left(I_n - C(e^{j\omega})\right)^{-*} \\
\Rightarrow\; & -\left(I_n - C(e^{j\omega})\right)^{-1} - \left(I_n - C(e^{j\omega})\right)^{-*} + I_n > 0 \\
\Rightarrow\; & \underbrace{\left[I_n - \left(I_n - C(e^{j\omega})\right)^{-1}\right]}_{\triangleq B(e^{j\omega})} + \left[I_n - \left(I_n - C(e^{j\omega})\right)^{-1}\right]^* > I_n > 0.
\end{aligned}$$

We thus have

$$\begin{cases} B(z) + B^*(z^{-*}) > 0, \quad \forall |z| = 1 \\ B(z) \text{ is causal} \end{cases}.$$

In other words, $B(z)$ is causal and positive real. Lemma 11 therefore implies that $B^{-1}(z)$ is causal. But since

$$C^{-1}(z) = [I_n - (I_n - C(z))]^{-1} = I_n - \left[I_n - (I_n - C(z))^{-1}\right]^{-1} = I_n - B^{-1}(z),$$

we conclude that $C^{-1}(z)$ is causal.

■

Lemma 11 *Suppose that the $n \times n$ transfer matrix $B(z)$ is causal and positive real,* i.e.,

(i) $B(z)$ is analytic on $|z| \geq 1$.

(ii) $B(z) + B^*(z^{-*}) > 0$ for $|z| = 1$.

Then $B^{-1}(z)$ is causal.

Proof: Since $B(z)$ is analytic on $|z| \geq 1$, then $B(e^{j\omega})$ is bounded for all $\omega \in [0, 2\pi]$ and, in particular,

$$\sup_{\omega \in [0,2\pi]} \bar{\sigma}\left[B(e^{j\omega})B^*(e^{j\omega})\right] = \|B(z)\|_\infty^2 < \infty.$$

Moreover, since $B(z)$ is positive real, there exists some $\alpha > 0$ such that

$$B(e^{j\omega}) + B^*(e^{j\omega}) > \alpha \|B(z)\|_\infty^2 I \geq \alpha B(e^{j\omega})B^*(e^{j\omega}), \quad \forall \omega \in [0, 2\pi].$$

Therefore the causal transfer matrix $I - \alpha B(z)$ is a strict contraction since

$$\begin{aligned} \left(I - \alpha B(e^{j\omega})\right)\left(I - \alpha B(e^{j\omega})\right)^* &= I - \alpha\left(B(e^{j\omega}) + B^*(e^{j\omega})\right) + \alpha^2 B(e^{j\omega})B^*(e^{j\omega}) \\ &= I - \alpha\left[\underbrace{B(e^{j\omega}) + B^*(e^{j\omega}) - \alpha B(e^{j\omega})B^*(e^{j\omega})}_{>0}\right] < I. \end{aligned}$$

Therefore, using Lemma 9, $B^{-1}(z)$ is causal since

$$B(z) = \frac{1}{\alpha} \cdot \left[I - (I - \alpha B(z))\right].$$

■

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