A Krein Space Interpretation of the Kalman-Yakubovich-Popov Lemma[1]

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Abstract

In this note we give a Krein space interpretation of the celebrated Kalman-Yakubovich-Popov (KYP) Lemma by introducing state-space models driven by inputs that lie in an indefinite-metric space. Such state-space models can be considered as generalizations of standard stochastic state-space models driven by stationary stochastic processes (that lie in a definite, or so-called Hilbert, space). In this framework, the KYP lemma corresponds to a certain decomposition in Krein space.

1 Introduction

The Kalman-Yakubovich-Popov (KYP) Lemma was first introduced and proven in [1, 2, 3] in the context of control theory. It is also closely related to passive network synthesis [4, 5], to dissipative dynamical systems [6] and to the stochastic realization problem [7]. In this note, we shall consider the KYP Lemma from a stochastic viewpoint.

In order to better understand the meaning of the KYP Lemma in the stochastic context, we introduce the concept of state-space models driven by inputs that lie in an indefinite, or so-called Krein, space [8]. Such state-space models are generalizations of standard stochastic state-space models driven by stationary stochastic processes, since in the former case the inputs lie in a definite (Hilbert) space. Within this framework, we obtain a characterization of the degrees of freedom in representing stationary stochastic processes with rational power spectral density functions, as well as a simple geometric interpretation of the KYP Lemma in terms of a certain decomposition in a Krein space.

To close the introduction, we remark that the results presented here describe the discrete-time scenario. Similar results hold for the continuous-time case, as well.

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2 Stationary Stochastic Processes and the Popov Function

Consider the time-invariant state-space model

$$\left\{ \begin{array}{rcl} \mathbf{x}_{i+1} & = & F\mathbf{x}_i + \mathbf{u}_i \\ \mathbf{y}_i & = & H\mathbf{x}_i + \mathbf{v}_i \end{array} \right. \tag{1}$$

where F is stable, $\{F, H\}$ is observable[3] and the disturbances are zero-mean *stationary* random processes with

$$E \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_j^* & \mathbf{v}_j^* & 1 \end{bmatrix} = \begin{bmatrix} Q\delta_{ij} & S\delta_{ij} & 0 \\ S^*\delta_{ij} & R\delta_{ij} & 0 \end{bmatrix}.$$

Taking z-transforms, we can rewrite (1) as

$$\mathbf{y}(z) = H(zI-F)^{-1}\mathbf{u}(z) + \mathbf{v}(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} \mathbf{u}(z) \\ \mathbf{v}(z) \end{bmatrix}.$$

Recall that if a $m \times 1$ stationary process $\{\mathbf{r}_i\}$ with z-spectral density function $S_r(z)$ is applied to a $p \times m$ linear system with transfer matrix $H(z)$ to yield an output $\{\mathbf{s}_i\}$, the so-called output z-spectrum defined as $S_s(z) \triangleq \mathcal{Z}\left\{E\mathbf{s}_j\mathbf{s}_{j-i}^*\right\}$, is given by

$$S_s(z) = H(z)S_r(z)H^*(z^{-*}).$$

Thus, in our case, the output z-spectrum of $\{\mathbf{y}_i\}$ is given by

$$S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{2}$$

Note that the matrix appearing in the center of (2) is the covariance of the disturbances $\{\mathbf{u}_i, \mathbf{v}_i\}$, so that we have

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0. \tag{3}$$

This implies that $S_y(e^{j\omega}) \geq 0$, which is the defining property of a power spectral density matrix generated by a true stochastic process.

However, let us calculate the output spectrum in an alternative fashion. The steady-state covariance of the state $\mathbf{x}_i$, defined by $\bar{\Pi} = \lim_{i\to\infty} E\mathbf{x}_i\mathbf{x}_i^*$, satisfies the (discrete-time) Lyapunov equation

$$\bar{\Pi} = F\bar{\Pi}F^* + Q. \tag{4}$$

[3] Both these conditions can be replaced with the less restrictive condition that $\{F, H\}$ is detectable, though for simplicity we shall continue to assume them.

Thus, in the steady-state, the autocorrelation function of the output is given by

$$R_{y,i} = E\mathbf{y}_j\mathbf{y}_{j-i}^* = \begin{cases} HF^i\bar{\Pi}H^* + HF^{i-1}S & i > 0 \\ R + H\bar{\Pi}H^* & i = 0 \\ H\bar{\Pi}F^{*i}H^* + S^*F^{*(i-1)}H^* & i < 0 \end{cases}$$

Taking the z-transform of $R_{y,i}$ in the above expression, the output z-spectrum can be written as

$$S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & F\bar{\Pi}H^* + S \\ H\bar{\Pi}F^* + S^* & R + H\bar{\Pi}H^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{5}$$

Comparing (2) with (5) we see that the only difference between these two *representations* of the output z-spectrum is the matrix appearing in the center of these equations. In the case of (2) we saw that this matrix was the covariance of the disturbances $\{\mathbf{u}_i, \mathbf{v}_i\}$. Now in the case of (5) the center matrix

$$\begin{bmatrix} 0 & F\bar{\Pi}H^* + S \\ H\bar{\Pi}F^* + S^* & R + H\bar{\Pi}H^* \end{bmatrix}, \tag{6}$$

is indefinite. Note that $S_y(e^{j\omega}) \geq 0$, of course, even though the center matrix (6) is not non-negative definite and cannot be thought of as the covariance of some random variables, say $\{\mathbf{u}_i^{(1)}, \mathbf{v}_i^{(1)}\}$. (Indeed $\mathbf{u}_i^{(1)}$ would need to have zero variance but nonzero cross-variance with $\mathbf{v}_i^{(1)}$!) However, if we broaden our domain of discourse, and instead of random variables, consider disturbances $\{\mathbf{u}_i, \mathbf{v}_i\}$ that belong to an abstract *indefinite* (so-called Krein) space, then the matrix (6) can be considered as the covariance of such an abstract disturbance $\{\mathbf{u}_i^{(1)}, \mathbf{v}_i^{(1)}\}$.[4]

The above discussion shows that even when considering state-space models driven by random variable disturbances (that lie in a Hilbert space) it is natural to consider indefinite metric spaces. Indeed there is much more to be gained from this generalization. Thus we shall gain an understanding of the fact that several different *center matrices* (*e.g.* those in (3) and (6)) can give rise to the same output z-spectrum.

2.1 An Equivalence Class for Input Covariances

To this end, consider the state-space model (1) but now suppose that the inputs $\{\mathbf{u}_i, \mathbf{v}_i\}$ are such that

$$\langle \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix} \rangle = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \delta_{ij}. \tag{7}$$

[4]We shall state what is exactly meant by a Krein space in Sec. 4.1. For the time being, it suffices to know that in a Krein space the variables $\{\mathbf{u}_i, \mathbf{v}_i\}$ may have indefinite covariance matrices.

Note that we have replaced the notation $E\mathbf{u}_i\mathbf{v}_j^*$ with $\langle \mathbf{u}_i, \mathbf{v}_j \rangle$ since we are now considering the $\{\mathbf{u}_i, \mathbf{v}_i\}$ to live in an indefinite space so that the matrix appearing in (7) may be indefinite. Now associated with the state-space model (1) and the inputs (7), we may define the *Popov function*

$$S_y(z) \triangleq \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{8}$$

We can readily see that the Popov function is the generalization of the z-power spectral density function since $S_y(z) = \mathcal{Z}\{\langle \mathbf{y}_j, \mathbf{y}_{j-i} \rangle\}$.

Now suppose that we intend to add white and stationary disturbances $\{\bar{\mathbf{u}}_i, \bar{\mathbf{v}}_i\}$ (orthogonal to the original $\{\mathbf{u}_i, \mathbf{v}_i\}$) to the state-space model (1) such that the output z-spectrum $S_y(z)$ remains unchanged. In other words, the output of the state-space model

$$\begin{cases} \mathbf{x}_{i+1} + \bar{\mathbf{x}}_{i+1} &= F(\mathbf{x}_i + \bar{\mathbf{x}}_i) + \mathbf{u}_i + \bar{\mathbf{u}}_i \\ \mathbf{y}_i + \bar{\mathbf{y}}_i &= H(\mathbf{x}_i + \bar{\mathbf{x}}_i) + \mathbf{v}_i + \bar{\mathbf{v}}_i \end{cases} \tag{9}$$

should still have Popov function equal to $S_y(z)$, given in (8).

The covariance matrix of the new disturbances $\mathbf{u}_i + \bar{\mathbf{u}}_i, \mathbf{v}_i + \bar{\mathbf{v}}_i$ is given by

$$\begin{bmatrix} Q+\bar{Q} & S+\bar{S} \\ S^*+\bar{S}^* & R+\bar{R} \end{bmatrix},$$

and the output z-spectrum by

$$S_{y+\bar{y}}(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q+\bar{Q} & S+\bar{S} \\ S^*+\bar{S}^* & R+\bar{R} \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}.$$

Now by linearity, $S_{y+\bar{y}}(z) = S_y(z) + S_{\bar{y}}(z)$. Therefore if $S_y(z)$ is to be unchanged, this implies that $S_{\bar{y}}(z)$, the z-spectrum of the proces $\{\bar{\mathbf{y}}_i\}$ defined by

$$\begin{cases} \bar{\mathbf{x}}_{i+1} &= F\bar{\mathbf{x}}_i + \bar{\mathbf{u}}_i \\ \bar{\mathbf{y}}_i &= H\bar{\mathbf{x}}_i + \bar{\mathbf{v}}_i \end{cases}, \tag{10}$$

must be *zero*. Now a simple calculation shows that

$$\langle \bar{\mathbf{y}}_i, \bar{\mathbf{y}}_i \rangle = \bar{R} + H\langle \bar{\mathbf{x}}_i, \bar{\mathbf{x}}_i \rangle H^*, \tag{11}$$

so that if we define $Z = -\langle \bar{\mathbf{x}}_i, \bar{\mathbf{x}}_i \rangle$, (note that since the variables in (10) belong to an indefinite metric space, Z is in general indefinite) we may write

$$\langle \bar{\mathbf{y}}_i, \bar{\mathbf{y}}_i \rangle = \bar{R} - HZH^* = 0, \tag{12}$$

or $\bar{R} = HZH^*$. Likewise, a similar computation for $i > j$, shows that

$$\langle \bar{\mathbf{y}}_i, \bar{\mathbf{y}}_j \rangle = HF^{i-j-1}(F\langle \bar{\mathbf{x}}_i, \bar{\mathbf{x}}_i \rangle H^* + \bar{S}) = HF^{i-j-1}(-FZH^* + \bar{S}). \tag{13}$$

Thus choosing

$$\bar{S} = FZH^* \tag{14}$$

we see that $\langle \bar{\mathbf{y}}_i, \bar{\mathbf{y}}_j \rangle = 0$. Finally, using the state equation in (10) we may write

$$-Z = -FZF^* + \bar{Q}. \tag{15}$$

Combining (12), (14) and (15) shows that the indefinite variables $\{\bar{\mathbf{u}}_i, \bar{\mathbf{v}}_i\}$ must have as covariance matrix

$$\begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^* & \bar{R} \end{bmatrix} = \begin{bmatrix} -Z + FZF^* & FZH^* \\ HZF^* & HZH^* \end{bmatrix}, \tag{16}$$

for some Hermitian Z (which is the negative of the steady state covariance matrix of the process $\bar{\mathbf{x}}_i$). We can thus show the following result.

Lemma 1 (Equivalence Class for Input Covariances) *(a) For any Hermitian Z, the output z-spectrum of the state-space model (1)*

$$S_y(z) = \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

is invariant under the input covariance transformation

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \rightarrow \begin{bmatrix} Q - Z + FZF^* & S + FZH^* \\ S^* + HZF^* & R + HZH^* \end{bmatrix}. \tag{17}$$

(b) If for an observable system $\{F, H\}$, there exist input covariances $\{Q_1, S_1, R_1\}$ and $\{Q_2, S_2, R_2\}$ that yield the same output spectrum, i.e.,

$$\begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_1 & S_1 \\ S_1^* & R_1 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}$$
$$= \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_2 & S_2 \\ S_2^* & R_2 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}$$

then there exists a <u>unique</u> Hermitian Z such that

$$\begin{bmatrix} Q_1 & S_1 \\ S_1^* & R_1 \end{bmatrix} = \begin{bmatrix} Q_2 - Z + FZF^* & S_2 + FZH^* \\ S_2^* + HZF^* & R_2 + HZH^* \end{bmatrix}.$$

Remark: When $\{F, H\}$ is not observable, part (b) of the above Lemma becomes slightly more complicated (see [9]). Although the results presented below extend to the case where $\{F, H\}$ is detectable instead of observable, to simplify the arguments we shall retain the observability assumption.

Proof of Lemma 1: We have already proven part (a) in the arguments preceding the statement of the Lemma. For part (b), note that since $\{Q_1, R_1, S_1\}$ and $\{Q_2, R_2, S_2\}$ generate the same output z-spectrum we can write

$$\begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_1 - Q_2 & S_1 - S_2 \\ S_1^* - S_2^* & R_1 - R_2 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} = 0.$$

Thus if we define the indefinite variables $\{\bar{\mathbf{u}}_i, \bar{\mathbf{v}}_i\}$ such that

$$\langle \begin{bmatrix} \bar{\mathbf{u}}_i \\ \bar{\mathbf{v}}_i \end{bmatrix}, \begin{bmatrix} \bar{\mathbf{u}}_j \\ \bar{\mathbf{v}}_j \end{bmatrix} \rangle = \begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^* & \bar{R} \end{bmatrix} \delta_{ij} \triangleq \begin{bmatrix} Q_1 - Q_2 & S_1 - S_2 \\ S_1^* - S_2^* & R_1 - R_2 \end{bmatrix} \delta_{ij},$$

then the state-space model

$$\begin{cases} \bar{\mathbf{x}}_{i+1} & = & F\bar{\mathbf{x}}_i + \bar{\mathbf{u}}_i \\ \bar{\mathbf{y}}_i & = & H\bar{\mathbf{x}}_i + \bar{\mathbf{v}}_i \end{cases},$$

must generate zero output z-spectrum. Using the arguments presented before the statement of the Lemma, this implies that

$$\begin{cases} \bar{Q} & = & -Z + FZF^* \\ \bar{R} & = & HZH^* \\ 0 & = & F^{i-j-1}(-FZH^* + \bar{S}) \quad \text{for } i > j \end{cases},$$

where as before we have defined $\langle \bar{\mathbf{x}}_i, \bar{\mathbf{x}}_i \rangle = -Z$. Note that since F is stable the first of the above equations shows that Z is unique.[5] Moreover, the last equation shows that

$$\mathcal{O}(-FZH^* + \bar{S}) = 0,$$

where $\mathcal{O} \triangleq \begin{bmatrix} H^* & F^*H^* & F^{2*}H^* & \dots \end{bmatrix}^*$ is the observability map. When $\{F, H\}$ is observable, $\mathcal{O}$ is full rank and we conclude that

$$-FZH^* + \bar{S} = 0.$$

We have thus shown that there exists a unique Hermitian Z such that

$$\begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^* & \bar{R} \end{bmatrix} = \begin{bmatrix} Q_1 - Q_2 & S_1 - S_2 \\ S_1^* - S_2^* & R_1 - R_2 \end{bmatrix} = \begin{bmatrix} -Z + FZF^* & FZH^* \\ HZF^* & HZH^* \end{bmatrix}.$$

■

[5]In fact, we only require that F have no two eigenvalues such that $\lambda_i = \lambda_j^{-*}$ for the solution to the Lyapunov equation $Z = FZF^* + \bar{Q}$ to be unique.

3 The KYP Lemma

The previous section showed the great freedom that is obtained by allowing the disturbances $\{\mathbf{u}_i, \mathbf{v}_i\}$ to have an indefinite covariance matrix. We were thus able to parametrize all input covariance matrices that gave rise to the same Popov function in terms of a Hermitian matrix Z. This matrix had the interpretation of being the steady state covariance of the state vector in a state-space model that generates zero output spectrum. The reader at this point may want to verify that the choice $Z = \bar{\Pi}$, where $\bar{\Pi}$ is as in (4), relates the input covariances in (2) and (5).

Another application of the degree of freedom available via the matrix Z, is to choose Z such that the center matrix in the Popov function drops rank, *i.e.*,

$$\begin{bmatrix} Q - Z + FZF^* & S + FZH^* \\ S^* + HZF^* & R + HZH^* \end{bmatrix} = \begin{bmatrix} K_p \\ I \end{bmatrix} R_e \begin{bmatrix} K_p^* & I \end{bmatrix}. \tag{18}$$

This is of significance, since it leads to the following factorization of the Popov function

$$S_y(z) = \left[H(zI - F)^{-1}K_p + I\right] R_e \left[H(z^{-*}I - F)^{-1}K_p + I\right]^*. \tag{19}$$

In particular, when the transfer matrix $H(zI-F)^{-1}K_p + I$ has a stable inverse, the above factorization is known as the canonical, or spectral, factorization of the Popov function. Although studying canonical factorizations of the Popov function lies beyond the scope of this note, we should remark that the above approach can be used to study solutions of the discrete-time algebraic Riccati equation (DARE) in terms of factorizations of the Popov function and has the benefit of treating the indefinite and positive (semi)definite cases in a unified fashion [9].

The results of Lemma 1 are not concerned with the case where the process $\{\mathbf{y}_i\}$ is a true stochastic process, *i.e.*, that its z-spectrum, $S_y(z)$, is nonnegative on the unit circle. When that is true, we have a further characterization of the Hermitian matrices, Z. The result is the KYP Lemma.

Theorem 1 (KYP Lemma) *Consider the observable pair $\{F, H\}$. Then the following two statements are equivalent:*

(i) $S_y(z) \geq 0$ *for all* $z = e^{j\omega} \notin \lambda(F)$, *where*

$$S_y(z) = \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}.$$

(ii) There exists a Hermitian Z such that

$$\begin{bmatrix} Q - Z + FZF^* & S + FZH^* \\ S^* + HZF^* & R + HZH^* \end{bmatrix} \geq 0. \tag{20}$$

Remark: In view of Lemma 1, we may replace the center matrix in $S_y(z)$ with the center matrix in (20). Thus, Theorem 1 has the following remarkable interpretation: $S_y(z)$ is a true z-spectral density function if, and only if, there exists true stochastic inputs with nonnegative definite covariance matrix that generate it! This has special significance to the problem of stochastic realization since it states that any nonnegative definite rational z-spectral density function can be realized by a finite-dimensional state-space model driven by true stochastic processes.

Although it is also possible to use Lemma 1, along with a factorization result of Youla [11], to prove the KYP lemma (this is similar to the proof given in [12]), we will instead focus on whether the KYP lemma admits further geometric interpretation in terms of Krein spaces.

4 A Geometric Interpretation

Let us begin with some definitions and notations. For a more complete discussion see [8, 10].

4.1 On Krein Spaces

Definition 1 (Krein Spaces) *An abstract vector space $\{\mathcal{K}, \langle\cdot,\cdot\rangle\}$ that satisfies the following requirements is called a Krein Space:*

(i) $\mathcal{K}$ is a linear space over $\mathcal{C}$, the field of complex numbers.

(ii) There exists a bilinear form $\langle\cdot,\cdot\rangle \in \mathcal{C}$ on $\mathcal{K}$ such that

(a) $\langle \mathbf{y}, \mathbf{x} \rangle \;=\; \langle \mathbf{x}, \mathbf{y} \rangle^*$

(b) $\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle \;=\; a\langle \mathbf{x}, \mathbf{z} \rangle \;+\; b\langle \mathbf{y}, \mathbf{z} \rangle$

for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{K}$, $a, b \in \mathcal{C}$, and where $*$ denotes complex conjugation.

(iii) The vector space $\mathcal{K}$ admits a direct orthogonal sum decomposition

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$$

such that $\{\mathcal{K}_+, \langle\cdot,\cdot\rangle\}$ and $\{\mathcal{K}_-, -\langle\cdot,\cdot\rangle\}$ are Hilbert spaces, and $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, for any $\mathbf{x} \in \mathcal{K}_+$ and $\mathbf{y} \in \mathcal{K}_-$.

Remarks:

1. Hilbert spaces satisfy not only (i) and (ii)-(a), (ii)-(b) above, but also the requirement that $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ when $\mathbf{x} \neq 0$.

2. The fundamental decomposition of $\mathcal{K}$ defines two projection operators $\mathcal{P}_+$ and $\mathcal{P}_-$ such that $\mathcal{P}_+\mathcal{K} = \mathcal{K}_+$ and $\mathcal{P}_-\mathcal{K} = \mathcal{K}_-$. Therefore for every $\mathbf{x} \in \mathcal{K}$ we can write

$$\mathbf{x} = P_+\mathbf{x} + P_-\mathbf{x} = \mathbf{x}_+ + \mathbf{x}_- \quad , \quad \mathbf{x}_\pm \in \mathcal{K}_\pm .$$

   Note that for every $\mathbf{x} \in \mathcal{K}_+$, we have $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, but the converse is not true: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ does not necessarily imply that $\mathbf{x} \in \mathcal{K}_+$.

3. A vector $\mathbf{x} \in \mathcal{K}$ will be said to be *positive* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, *neutral* if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, or *negative* if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$. Correspondingly, a subspace $\mathcal{M} \subset \mathcal{K}$ can be positive, neutral, or negative, if all its elements are so, respectively.

Some geometric insight into Krein spaces may be gained by considering the special 3-dimensional, so-called Minkowski, space of Figure 1, defined by the inner product

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = x_1 x_2 + y_1 y_2 - t_1 t_2$$

where $\mathbf{v}_1 = (x_1, y_1, t_1)$, $\mathbf{v}_2 = (x_2, y_2, t_2)$, and $x_i, y_i, t_i \in \mathcal{C}$. The (indefinite) squared norm of each vector $\mathbf{v} = (x, y, t)$ is equal to

$$\langle \mathbf{v}, \mathbf{v} \rangle = x^2 + y^2 - t^2.$$

In this case, we can take $\mathcal{K}_+$ to be the $x - y$ plane and $\mathcal{K}_-$ as the t axis. The neutral subspace is given by the cone, $x^2 + y^2 - t^2 = 0$, with points inside the cone belonging to the negative subspace and points outside the cone corresponding to the positive subspace.

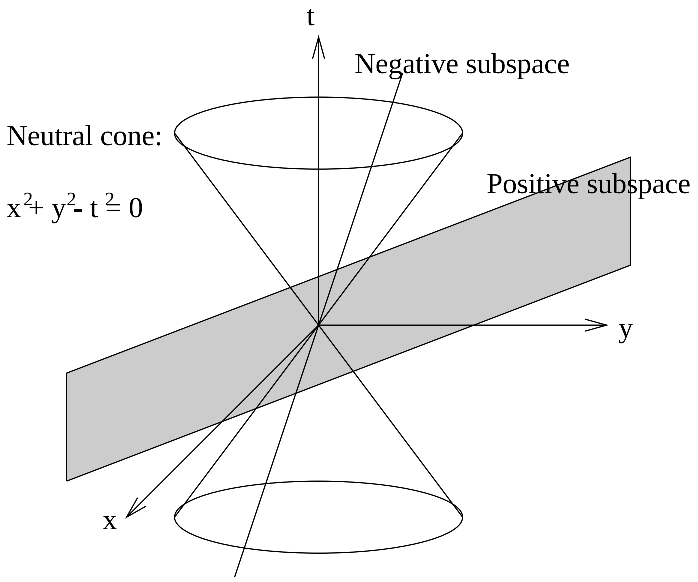

Figure 1: 3-dimensional Minkowski space

Finally, it will be useful to introduce a slight generalization of the definition of Krein spaces that has been given so far. In Definition 1, we mentioned that $\mathcal{K}$ should be linear over the field of complex numbers, $\mathcal{C}$. However, it turns out that we can replace $\mathcal{C}$ with any ring $\mathcal{S}$. The only difference in the first two axioms is that the operation $*$ is now an *involution* that depends on the ring $\mathcal{S}$.

When the inner product $\langle\cdot,\cdot\rangle \in \mathcal{S}$ is positive, $\{\mathcal{K}, \langle\cdot,\cdot\rangle\}$ is referred to as a Hilbert *module*. Thus the third axiom for Krein spaces can be replaced by

(iii) The vector space $\mathcal{K}$ admits a direct orthogonal sum decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ such that $\{\mathcal{K}_+, \langle\cdot,\cdot\rangle\}$ and $\{\mathcal{K}_-, -\langle\cdot,\cdot\rangle\}$ are Hilbert modules, and $\langle \mathbf{x}, \mathbf{y}\rangle = 0$ for any $\mathbf{x} \in \mathcal{K}_+$ and $\mathbf{y} \in \mathcal{K}_-$.

4.2 The Geometric Setup

Since our interpretation of the KYP Lemma will use Krein space geometry, we begin by carefully defining the spaces in which our inputs and outputs lie.

The input space, $\mathcal{K}^{(d)}$, consists of all white input sequences $\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \left\{ \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \right\}_{i=-\infty}^{\infty}$ with some (possibly indefinite) arbitrary covariance matrix $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$. Thus, we may write

$$\mathcal{K}^{(d)} = \left\{ \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \left\{ \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \right\}_{i=-\infty}^{\infty} \middle| \langle \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix} \rangle = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \delta_{ij} \right\}. \tag{21}$$

We shall now define the inner product in $\mathcal{K}^{(d)}$ to be

$$\langle \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{v}^1 \end{bmatrix}, \begin{bmatrix} \mathbf{u}^2 \\ \mathbf{v}^2 \end{bmatrix} \rangle = \mathcal{F}\left\{ \langle \begin{bmatrix} \mathbf{u}_j^1 \\ \mathbf{v}_j^1 \end{bmatrix}, \begin{bmatrix} \mathbf{u}_{j-i}^2 \\ \mathbf{v}_{j-i}^2 \end{bmatrix} \rangle \right\}, \tag{22}$$

where the notation $\mathcal{F}\{\cdot\}$ denotes the discrete Fourier transform. The inner product is thus a matrix rational function of $e^{j\omega}$, and $\mathcal{K}^{(d)}$ is a Krein space over the ring of such matrix rational functions. It is straightforward to see that $\mathcal{K}_+^{(d)}$ and $\mathcal{K}_-^{(d)}$ have the obvious structure

$$\mathcal{K}_+^{(d)} = \left\{ \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \left\{ \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \right\}_{i=-\infty}^{\infty} \middle| \langle \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix} \rangle = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \delta_{ij} \geq 0 \right\}, \tag{23}$$

and

$$\mathcal{K}_-^{(d)} = \left\{ \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \left\{ \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \right\}_{i=-\infty}^{\infty} \middle| \langle \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix} \rangle = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \delta_{ij} \leq 0 \right\}. \tag{24}$$

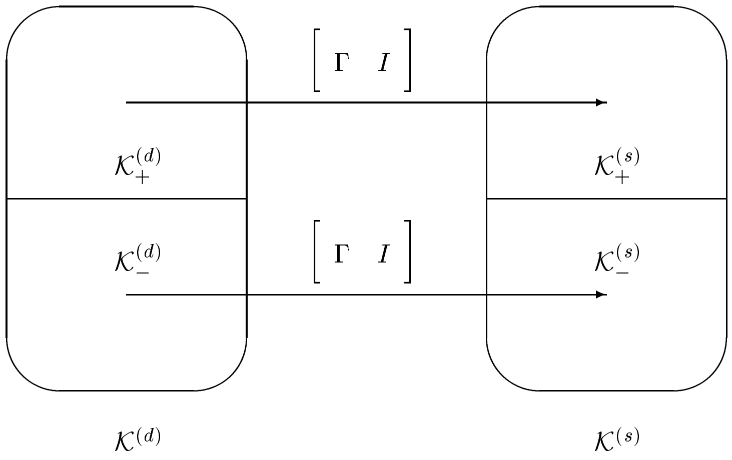

Figure 2: Mapping from input space to output space.

To now construct the output space, consider the time-invariant state-space model

$$\begin{cases} \mathbf{x}_{i+1} &= F\mathbf{x}_i + \mathbf{u}_i \\ \mathbf{y}_i &= H\mathbf{x}_i + \mathbf{v}_i \end{cases} \tag{25}$$

where the $\{\mathbf{u}_i, \mathbf{v}_i\}$ lie in $\mathcal{K}^{(d)}$. Now if we define the output sequence, $\mathbf{y} = \{\mathbf{y}_i\}_{i=-\infty}^{\infty}$, we may formally write

$$\mathbf{y} = \Gamma\mathbf{u} + \mathbf{v} = \begin{bmatrix} \Gamma & I \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}, \tag{26}$$

where Γ and I denote the impulse response and identity maps, respectively. Thus, the $\mathbf{y}$ also form a Krein space, $\mathcal{K}^{(s)}$. We can formally represent this Krein space as

$$\mathbf{y} \in \mathcal{K}^{(s)} = \begin{bmatrix} \Gamma & I \end{bmatrix} \mathcal{K}^{(d)},$$

and in particular $\mathcal{K}^{(s)} = \mathcal{K}_+^{(s)} \oplus \mathcal{K}_-^{(s)}$, with

$$\mathcal{K}_+^{(s)} = \begin{bmatrix} \Gamma & I \end{bmatrix} \mathcal{K}_+^{(d)} \quad \text{and} \quad \mathcal{K}_-^{(s)} = \begin{bmatrix} \Gamma & I \end{bmatrix} \mathcal{K}_-^{(d)}. \tag{27}$$

Thus, $\mathcal{K}^{(s)}$ is the Krein space of all possible outputs of (25) when the inputs are from the Krein space $\mathcal{K}^{(d)}$, and $\mathcal{K}_+^{(s)}$ is the Hilbert space of all possible outputs of (25) when the inputs are from the Hilbert space $\mathcal{K}_+^{(d)}$. Likewise for $\mathcal{K}_-^{(s)}$. See Fig. 2.

Finally, we should remark that for any element, $\mathbf{y}$, of the Krein space $\mathcal{K}^{(s)}$ that is generated by the white sequence $\{\mathbf{u}, \mathbf{v}\}$ with covariance $\{Q, S, R\}$, we may write

$$\langle \mathbf{y}, \mathbf{y} \rangle = S_y(e^{j\omega}) = \begin{bmatrix} H(e^{j\omega}I - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (e^{-j\omega}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{28}$$

Thus the squared norm of $\mathbf{y}$ is the Popov function, $S_y(e^{j\omega})$.

4.3 A Simple Decomposition

In this section we shall develop a simple geometric interpretation of the KYP Lemma in terms of a certain decomposition in Krein space. Recall from the discussion at the end of Sec. 4.2 that the Popov function $S_y(z)$ can be regarded as the squared norm of $\mathbf{y}$, some element of the Krein space $\mathcal{K}^{(s)}$. Now the premise of the KYP Lemma is such that $S_y(e^{j\omega}) \geq 0$, *i.e.*, that $\mathbf{y}$ has positive squared norm and belongs to the positive subspace of $\mathcal{K}^{(s)}$.

Now using the KYP Lemma, choose a Hermitian Z such that

$$\begin{bmatrix} Q - Z + FZF^* & S + FZH^* \\ S^* + HZF^* & R + HZH^* \end{bmatrix} \geq 0.$$

Since this is a nonegative definite input covariance the output process $\mathbf{y}^+$ associated with it belongs to $\mathcal{K}_+^{(s)}$. Moreover, the output process, $\mathbf{y}^0 = \mathbf{y} - \mathbf{y}^+$, with input covariance

$$\begin{bmatrix} Z - FZF^* & -FZH^* \\ -HZF^* & -HZH^* \end{bmatrix},$$

is a neutral element of $\mathcal{K}^{(s)}$ since $\langle \mathbf{y}^0, \mathbf{y}^0 \rangle = 0$. The KYP Lemma obviously states

$$\langle \mathbf{y}, \mathbf{y} \rangle = \langle \mathbf{y}^+, \mathbf{y}^+ \rangle + \langle \mathbf{y}^0, \mathbf{y}^0 \rangle. \tag{29}$$

The following result is now straightforward.

Geometric Interpretation of the KYP Lemma *Consider an element* $\mathbf{y} \in \mathcal{K}^{(s)}$. *Then* $\mathbf{y}$ *has positive squared norm,* i.e., $\langle \mathbf{y}, \mathbf{y} \rangle \geq 0$, *if, and only if, it can be decomposed as follows*

$$\mathbf{y} = \mathbf{y}^+ + \mathbf{y}^0, \tag{30}$$

where $\mathbf{y}^+ \in \mathcal{K}_+^{(s)}$ *is such that* $\langle \mathbf{y}^+, \mathbf{y}^+ \rangle = \langle \mathbf{y}, \mathbf{y} \rangle \geq 0$, *and* $\mathbf{y}^0$ *is neutral,* i.e., $\langle \mathbf{y}^0, \mathbf{y}^0 \rangle = 0$. *Moreover, note that* $\langle \mathbf{y}^+, \mathbf{y}^0 \rangle = 0$, i.e., $\mathbf{y}^+$ *and* $\mathbf{y}^0$ *are orthogonal.*

To gain further insight into the above decomposition of $\mathbf{y}$, let us write the (unique) fundamental decomposition of the elements $\mathbf{y}$ and $\mathbf{y}^0$ into their components in $\mathcal{K}_+^{(s)}$ and $\mathcal{K}_-^{(s)}$, *i.e.*,

$$\mathbf{y} = \mathbf{y}_+ + \mathbf{y}_- \quad \text{and} \quad \mathbf{y}^0 = \mathbf{y}_+^0 + \mathbf{y}_-^0,$$

where $\mathbf{y}_+, \mathbf{y}_+^0 \in \mathcal{K}_+^{(s)}$ and $\mathbf{y}_-, \mathbf{y}_-^0 \in \mathcal{K}_+^{(s)}$. Therefore, using (30) we may write

$$\mathbf{y}_+ + \mathbf{y}_- = \mathbf{y}^+ + (\mathbf{y}_+^0 + \mathbf{y}_-^0).$$

Equating the components of the above equality that belong to $\mathcal{K}_+^{(s)}$ and $\mathcal{K}_-^{(s)}$, respectively, yields

$$\mathbf{y}_+ = \mathbf{y}^+ + \mathbf{y}_+^0, \tag{31}$$

and $\mathbf{y}_- = \mathbf{y}^0_-$. Eq. (31) has a very interesting interpretation. First note that since $\mathbf{y}^+$ is othogonal to both $\mathbf{y}^0$ and $\mathbf{y}^0_-$ it must be othogonal to $\mathbf{y}^0_+$ as well. Thus (31) is an orthogonal decomposition in $\mathcal{K}^{(s)}_+$: it shows that the given element $\mathbf{y}_+$ with squared norm larger than the squared norm of $\mathbf{y}$, *i.e.*, $\langle \mathbf{y}_+, \mathbf{y}_+ \rangle \geq \langle \mathbf{y}, \mathbf{y} \rangle \geq 0$, can be orthogonally decomposed into two elements, $\mathbf{y}^+$ and $\mathbf{y}^0_+$, one of which has squared norm equal to the squared norm of $\mathbf{y}$, *i.e.*, $\langle \mathbf{y}^+, \mathbf{y}^+ \rangle = \langle \mathbf{y}, \mathbf{y} \rangle$. Roughly speaking, if we consider the hypersphere in $\mathcal{K}^{(s)}_+$ of radius $\langle \mathbf{y}, \mathbf{y} \rangle \geq 0$, then $\mathbf{y}^+$ is obtained from drawing the tangent from $\mathbf{y}_+$ to this hypersphere (see Fig. 3).

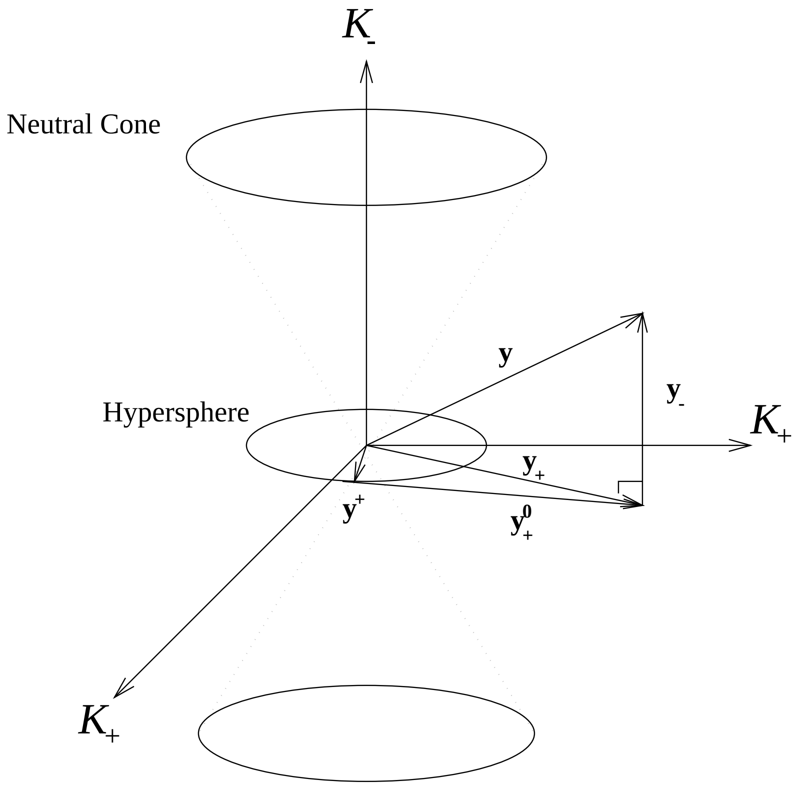

Figure 3: Decomposition of positive vectors

5 Conclusion

In this note we studied the KYP Lemma from a geometric and stochastic point of view. We did so, by introducing state-space models driven by inputs that lie in some indefinite (Krein) space and by studying their associated Popov function. Here we discussed the stationary version of the KYP Lemma. A time-varying version of this Lemma also exists and is studied in [10].

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